LIST OF SYMBOLS AND ABBREVIATIONS

Symbols

B	Viscous friction coefficient, N-sec/m
e_b	Back emf, v
e_{ss}	Steady state error
f	Applied force, N
f_m	Opposing force offered by mass of the body,N
f_k	Opposing force offered by the elasticity of the body, N
f_b	Opposing force offered by the friction of the body, N
F_s	Sampling frequency, Hz
G	Conductance, mho
H	Transformation or operator
i_a	Armature current, A
i_f	Field current, A
J	Moment of inertia, kg-m^2/rad
j	Complex operator
K	Stiffness of the spring, N-m / rad
K_a	Acceleration error constant
K_b	Back emf constant, V / (rad/sec)
K_d	Derivative constant or gain
K_i	Integral constant or gain
K_g	Gain Margin
K_m	Motor gain constant
K_p	Proportional gain
K_t	Torque constant, N-m/A
K_{tf}	Torque constant, Nm / A
K_v	Velocity error constant
L_a	Armature inductance, H

L_f	Field inductance, H
M	Mass, kg
M_p	Maximum overshoot
M_r	Resonant peak
n	Order of the system
N	Type number
p	Pole of a system
p_c	Pole of compensator
P_k	Forward path gain of K^{th} forward path
q	Charge
R_a	Armature resistance, Ω
R_f	Field resistance, Ω
s	Complex variable
s_d	Dominant pole
T	Applied torque, N-m
T_a	Electrical time constant
T_b	Opposing torque due to friction, N-m
t_d	Delay time
T_d	Derivative time
T_f	Field time constant
T_i	Integral time
T_j	Opposing torque due to moment of inertia, N-m
T_k	Opposing torque due to elasticity, N-m
T_m	Mechanical time constant
t_p	Peak time
t_r	Rise time
t_s	Settling time
u_b	Normalized bandwidth

u_r	Normalized resonant frequency
V_a	Armature voltage, V
V_f	Field voltage, V
x	Displacement, m
z	Zero of a system
z_c	Zero of compensator
θ	Angular displacement, rad
$\frac{d\theta}{dt}$	Angular velocity, rad/sec
$\frac{d^2\theta}{dt^2}$	Angular acceleration, rad/sec^2
ω_n	Undamped natural frequency, rad/sec
ζ	Damping ratio
ω_r	Resonant frequency
ω_b	Bandwidth
γ	Phase margin
ω_{pc}	Phase crossover frequency
ω_{gc}	Gain crossover frequency
ϕ	Flux, weber
ω_c	Corner frequency
ω_d	Damped frequency of oscillation
α	Phase angle
ω_m	Frequency of maximum phase lag/lead
ϕ_m	Maximum lag/lead angle
ε	Additional phase lead
ϕ_a	Angle of asymptotes
ϕ_p	Angle of departure
ϕ_z	Angle of arrival

λ	Eigen value
δ_T	Impulse train

Standard Input/Output signals

c(t)	Response in time domain
c(k)	Response of discrete signal
e(t)	Error signal
f(kT)	Digital error signal
g(kT)	Digital control signal
r(t)	Input in time domain
r(k)	Discrete time input signal
u(t)	Control signal (Analog)
$\delta(t)$	Impulse signal

Matrices and Vectors

A	System matrix
$\mathbf{A}^K$	State transition matrix of discrete system
B	Input matrix
C	Output matrix
D	Transmission matrix
e^{At}	State transition matrix
I	Identity matrix
J	Jordan matrix
M	Modal matrix or diagonalization matrix
$\mathbf{P}_0$ / $\mathbf{P}_c$	Transformation matrix
$\mathbf{Q}_c$	Composite matrix for controllability
$\mathbf{Q}_0$	Composite matrix for observability
U(t)	Input vector

U(k)	Input vector of discrete time system
V	Vander monde matrix
X(t)	State variable vector
$\mathbf{X}_o$	Initial condition vector
X(k)	State vector of discrete time system
Y(t)	Output vector
Y(k)	Output vector of discrete time system
$\wedge$	Grammian matrix

Transform Operators and Functions

A(s)	Auxiliary polynomial
E(s)	Error signal in s-domain
G(s)	Open loop transfer function
G(s)H(s)	Loop transfer function
H(s)	Feedback transfer function
$\mathcal{L}$	Laplace transform
$\mathcal{L}^{-1}$	Inverse Laplace transform
M(s)	Closed loop transfer function
T(s)	Transfer function of the system
$\mathcal{Z}$	$\mathcal{Z}$-transform
$\mathcal{Z}^{-1}$	Inverse $\mathcal{Z}$-transform

Abbreviations

BIBO	Bounded Input Bounded Output
LDS	Linear Discrete Time System
LTI	Linear Time Invariant System
ROC	Region of convergence
ZOH	Zero Order Hold

CHAPTER 1

MATHEMATICAL MODELS OF CONTROL SYSTEM

1.1 CONTROL SYSTEM

Control system theory evolved as an engineering discipline and due to universality of the principles involved, it is extended to various fields like economy, sociology, biology, medicine, etc. Control theory has played a vital role in the advance of engineering and science. The automatic control has become an integral part of modern manufacturing and industrial processes. For example, numerical control of machine tools in manufacturing industries, controlling pressure, temperature, humidity, viscosity and flow in process industry.

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a ***system.*** In a system when the output quantity is controlled by varying the input quantity, the system is called ***control system.*** The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

OPEN LOOP SYSTEM

Any physical system which does not automatically correct the variation in its output, is called an ***open loop system***, or control system in which the output quantity has no effect upon the input quantity are called open-loop control system. This means that the output is not fedback to the input for correction.

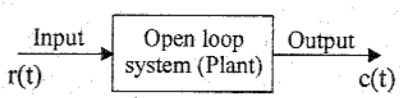


Fig 1.1 : *Open loop system.*

In open loop system the output can be varied by varying the input. But due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct the output. In open loop systems the changes in output are corrected by changing the input manually.

CLOSED LOOP SYSTEM

Control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called ***closed loop systems.***

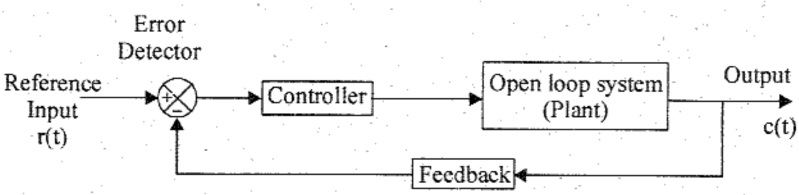


Fig 1.2 : *Closed loop system.*

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called ***automatic control system.*** The general block diagram of an automatic control system is shown in fig 1.2. It consists of an error detector, a controller, plant (open loop system) and feedback path elements.

The reference signal (or input signal) corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal is proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

Advantages of open loop systems

1. The open loop systems are simple and economical.
2. The open loop systems are easier to construct.
3. Generally the open loop systems are stable.

Disadvantages of open loop systems

1. The open loop systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not corrected automatically.

Advantages of closed loop systems

1. The closed loop systems are accurate.
2. The closed loop systems are accurate even in the presence of non-linearities.
3. The sensitivity of the systems may be made small to make the system more stable.
4. The closed loop systems are less affected by noise.

Disadvantages of closed loop systems

1. The closed loop systems are complex and costly.
2. The feedback in closed loop system may lead to oscillatory response.
3. The feedback reduces the overall gain of the system.
4. Stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

1.2 EXAMPLES OF CONTROL SYSTEMS

EXAMPLE 1 : TEMPERATURE CONTROL SYSTEM

OPEN LOOP SYSTEM

The electric furnace shown in fig 1.3. is an open loop system. The output in the system is the desired temperature. The temperature of the system is raised by heat generated by the heating element. The output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of the supply is governed by the time setting of the relay. The temperature is measured by a sensor, which gives an analog voltage corresponding to the temperature of the furnace. The analog signal is converted to digital signal by an Analog - to - Digital converter (A/D converter).

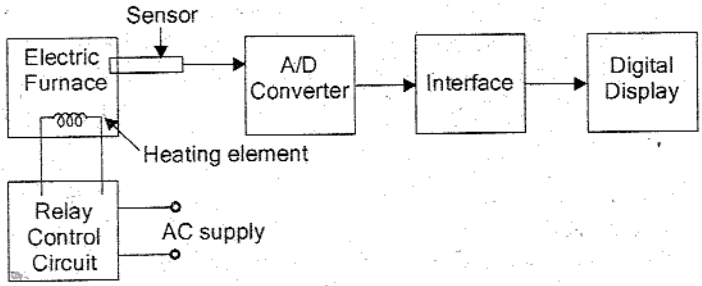


Fig 1.3 : Open loop temperature control system.

The digital signal is given to the digital display device to display the temperature. In this system if there is any change in output temperature then the time setting of the relay is not altered automatically.

CLOSED LOOP SYSTEM

The electric furnace shown in fig 1.4 is a closed loop system. The output of the system is the desired temperature and it depends on the time during which the supply to heater remains ON.

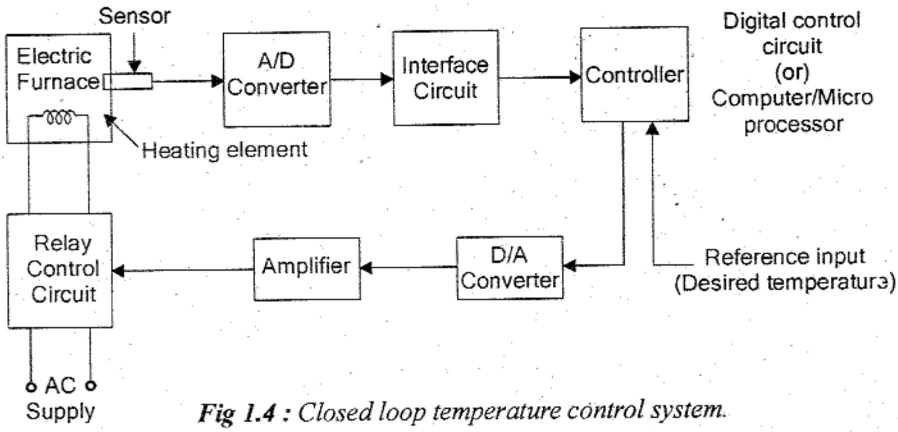


Fig 1.4 : Closed loop temperature control system.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The desired temperature is input to the system through keyboard or as a signal corresponding to desired temperature via ports. The actual temperature is sensed by sensor and converted to digital signal by the A/D converter. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output. Hence it is a closed loop system.

EXAMPLE 2 : TRAFFIC CONTROL SYSTEM

OPEN LOOP SYSTEM

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. Since the time slot does not changes according to traffic density, the system is open loop system.

CLOSED LOOP SYSTEM

Traffic control system can be made as a closed loop system if the time slots of the signals are decided based on the density of traffic. In closed loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic. Since the closed loop system dynamically changes the timings, the flow of vehicles will be better than open loop system.

EXAMPLE 3 : NUMERICAL CONTROL SYSTEM

OPEN LOOP SYSTEM

Numerical control is a method of controlling the motion of machine components using numbers. Here, the position of work head tool is controlled by the binary information contained in a disk.

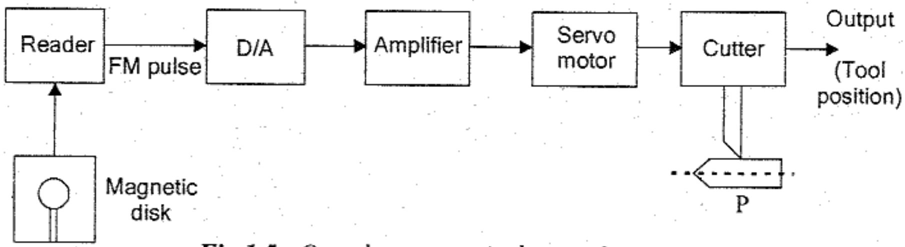


Fig 1.5 : Open loop numerical control system.

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). The tool will operate on the desired part P. To start the system, the disk is fed through the reader to the D/A converter. The D/A converter converts the FM(frequency modulated) output of the reader to a analog signal. It is amplified and fed to servometer which positions the cutter on the desired part P. The position of the cutter head is controlled by the angular motion of the servometer. This is an open loop system since no feedback path exists between the output and input. The system positions the tool for a given input command. Any deviation in the desired position is not checked and corrected automatically.

CLOSED LOOP SYSTEM

A magnetic disk is prepared in binary form representing the desired part P (P is the metal part to be machined). To start the system, the disk is loaded in the reader. The controller compares the frequency modulated input pulse signal with the feedback pulse signal. The controller is a computer or microprocessor system. The controller carries out mathematical operations on the difference in the pulse signals and generates an error signal. The D/A converter converts the controller output pulse (error signal) into an analog signal . The amplified analog signal rotates the servomotor to position the tool on the job. The position of the cutterhead is controlled according to the input of the servomotor.

The transducer attached to the cutterhead converts the motion into an electrical signal. The analog electrical signal is converted to the digital pulse signal by the A/D converter. Then this signal is compared with the input pulse signal. If there is any difference between these two, the controller sends a signal to the servomotor to reduce it. Thus the system automatically corrects any deviation in the desired output tool position. An advantage of numerical control is that complex parts can be produced with uniform tolerances at the maximum milling speed.

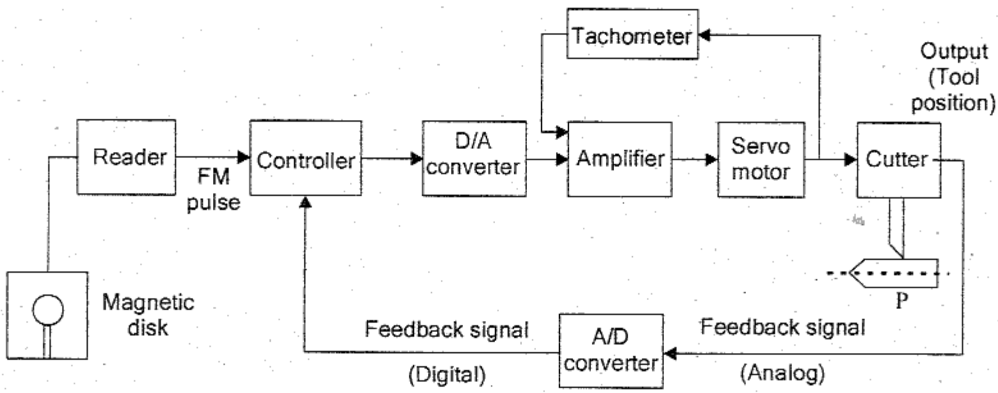


Fig 1.6 : Closed loop numerical control system.

EXAMPLE 4 : POSITION CONTROL SYSTEM USING SERVOMOTOR

The position control system shown in fig 1.7 is a closed loop system. The system consists of a servomotor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. Potentiometers are used to convert the mechanical motion to electrical signals. The desired load position (θ_R) is set on the input potentiometer and the actual load position (θ_c) is fed to feedback potentiometer. The difference between the two angular positions generates an error signal, which is amplified and fed to generator field circuit. The induced emf of the generator drives the motor. The rotation of the motor stops when the error signal is zero, i.e. when the desired load position is reached.

This type of control systems are called servomechanisms . The *servo* or *servomechanisms* are feedback control systems in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

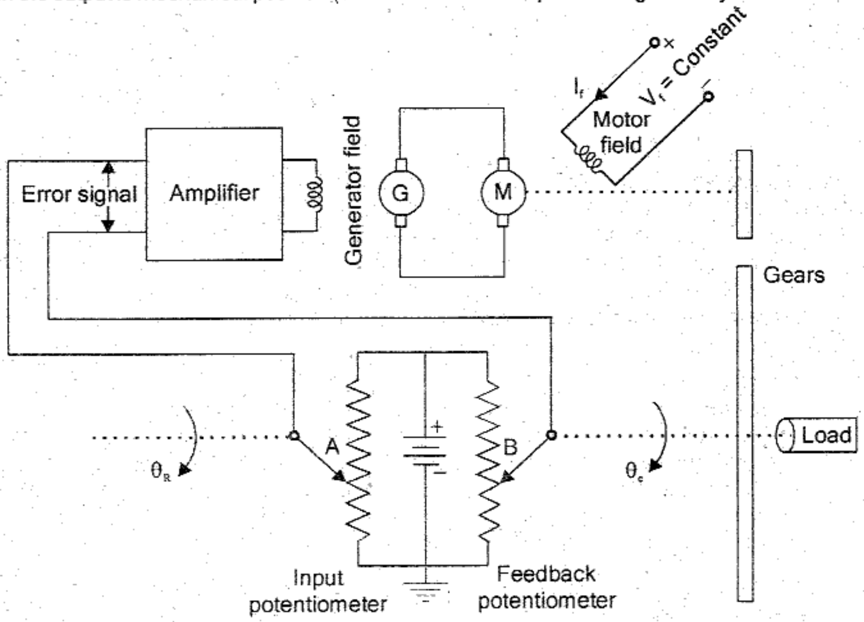


Fig 1.7 : A position control system (servomechanism).

1.3 MATHEMATICAL MODELS OF CONTROL SYSTEMS

A ***control system*** is a collection of physical objects (components) connected together to serve an objective. The input output relations of various physical components of a system are governed by ***differential equations.*** The mathematical model of a control system constitutes a set of differential equations. The response or output of the system can be studied by solving the differential equations for various input conditions.

The mathematical model of a system is linear if it obeys the principle of superposition and homogenity. This principle implies that if a system model has responses $y_1(t)$ and y_2 (t) to any inputs x_1 (t) and x_2 (t) respectively, then the system response to the linear combination of these inputs a_1x_1 (t) + a_2 x_2 (t) is given by linear combination of the individual outputs a_1 $y_1(t)+a_2$ $y_2(t)$, where a_1 and a_2 are constants.

The principle of superposition can be explained diagrammatically as shown in fig. 1.8.

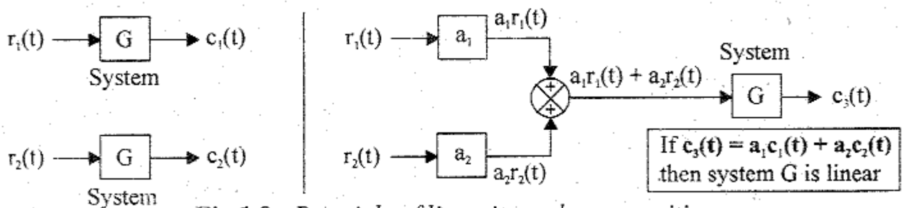


Fig 1.8 : Principle of linearity and superposition.

A mathematical model will be linear if the differential equations describing the system has constant coefficients (or the coefficients may be functions of independent variables). If the coefficients of the differential equation describing the system are constants then the model is ***linear time invariant.*** If the coefficients of differential equations governing the system are functions of time then the model is ***linear time varying.***

The differential equations of a linear time invariant system can be reshaped into different form for the convenience of analysis. One such model for single input and single output system analysis is transfer function of the system. The ***transfer function*** of a system is defined as the ratio of Laplace transform of output to the Laplace transform of input with zero initial conditions.

$$\boxed{\text{Transfer function} = \left.\frac{\text{Laplace Transform of output}}{\text{Laplace Transform of input}}\right|_{\text{with zero initial conditions}}} \quad .....(1.1)$$

The transfer function can be obtained by taking Laplace transform of the differential equations governing the system with zero initial conditions and rearranging the resulting algebraic equations to get the ratio of output to input.

1.4 MECHANICAL TRANSLATIONAL SYSTEMS

The model of mechanical translational systems can be obtained by using three basic elements ***mass, spring and dash-pot.*** These three elements represents three essential phenomena which occur in various ways in mechanical systems.

The weight of the mechanical system is represented by the element ***mass*** and it is assumed to be concentrated at the center of the body. The elastic deformation of the body can be represented by a ***spring.*** The friction existing in rotating mechanical system can be represented by the ***dash-pot.*** The dash-pot is a piston moving inside a cylinder filled with viscous fluid.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by ***Newton's second law of motion.*** For translational systems it states that the sum of forces acting on a body is zero. (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \dfrac{dx}{dt}$ = Velocity, m/sec

$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : Lower case letters are functions of time

FORCE BALANCE EQUATIONS OF IDEALIZED ELEMENTS

Consider an ideal mass element shown in fig 1.9 which has negligible friction and elasticity. Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body.

Let, f = Applied force

f_m = Opposing force due to mass

Here, $f_m \propto \dfrac{d^2x}{dt^2}$ or $f_m = M\dfrac{d^2x}{dt^2}$

By Newton's second law, $\boxed{f = f_m = M\dfrac{d^2x}{dt^2}}$ .....(1.2)

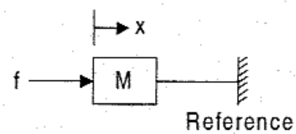


Fig 1.9 : Ideal mass element.

Consider an ideal frictional element dashpot shown in fig 1.10 which has negligible mass and elasticity. Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body.

Let, f = Applied force

f_b = Opposing force due to friction

Here, $f_b \propto \dfrac{dx}{dt}$ or $f_b = B\dfrac{dx}{dt}$

By Newton's second law, $\boxed{f = f_b = B\dfrac{dx}{dt}}$ .....(1.3)

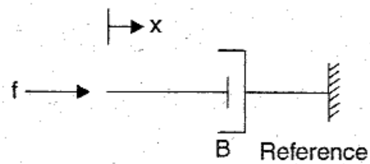


Fig 1.10 : Ideal dashpot with one end fixed to reference.

When the dashpot has displacement at both ends as shown in fig 1.11, the opposing force is proportional to differential velocity.

$$f_b \propto \frac{d}{dt}(x_1 - x_2) \qquad \text{or} \qquad f_b = B\frac{d}{dt}(x_1 - x_2)$$

$$\therefore \boxed{f = f_b = B\frac{d}{dt}(x_1 - x_2)} \quad .....(1.4)$$

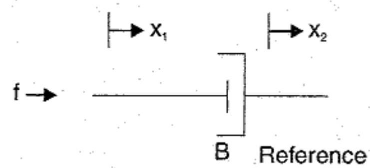


Fig 1.11 : Ideal dashpot with displacement at both ends.

Consider an ideal elastic element spring shown in fig 1.12, which has negligible mass and friction. Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body.

Let, f = Applied force

f_k = Opposing force due to elasticity

Here $f_k \propto x$ or $f_k = K\,x$

By Newton's second law, $\boxed{f = f_k = Kx}$ .....(1.5)

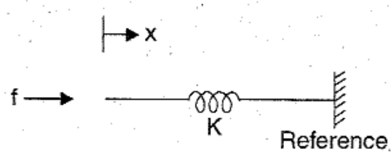


Fig 1.12 : Ideal spring with one end fixed to reference.

When the spring has displacement at both ends as shown in g 1.13 the opposing force is proportional to differential displacement.

$$f_k \propto (x_1 - x_2) \qquad \text{or} \qquad f_k = K(x_1 - x_2)$$

$$\therefore \boxed{f = f_k = K(x_1 - x_2)} \quad .....(1.6)$$

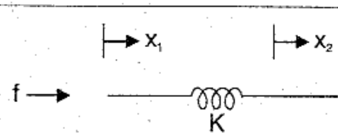


Fig 1.13 : Ideal spring with displacement at both ends.

Guidelines to determine the Transfer Function of Mechanical Translational System

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as x_1, x_2, x_3, etc., and assign a displacement to each mass(node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration.

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass (node). Always the opposing force acts in a direction opposite to applied force. The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

4. For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take Laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system.

Note : *Laplace transform of* $x(t) = \mathcal{L}\{x(t)\} = X(s)$

Laplace transform of $\dfrac{dx(t)}{dt} = \mathcal{L}\left\{\dfrac{d}{dt}\,x(t)\right\} = s\,X(s)$ *(with zero initial conditions)*

Laplace transform of $\dfrac{d^2x(t)}{dt^2} = \mathcal{L}\left\{\dfrac{d^2}{dt^2}\,x(t)\right\} = s^2\,X(s)$ *(with zero initial conditions)*

EXAMPLE 1.1

Write the differential equations governing the mechanical system shown in fig 1. and determine the transfer function.

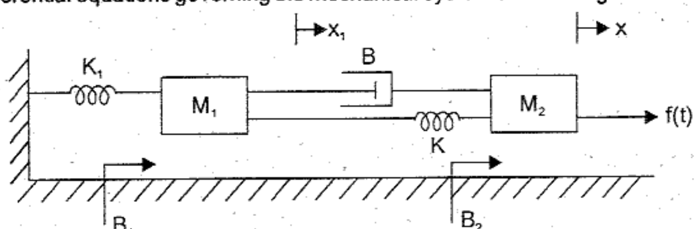


Fig 1.

SOLUTION

In the given system, applied force 'f(t)' is the input and displacement 'x' is the output.

Let, Laplace transform of f(t) $= \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of $x = \mathcal{L}\{x\} = X(s)$

Laplace transform of $x_1 = \mathcal{L}\{x_1\} = X_1(s)$

Hence the required transfer function is $\dfrac{X(s)}{F(s)}$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be x_1. The free body diagram of mass M_1 is shown in fig 2. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1} and f_k.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}\ ; \qquad f_{b1} = B_1 \frac{dx_1}{dt}\ ; \qquad f_{k1} = K_1 x_1;$$

$$f_b = B\frac{d}{dt}(x_1 - x)\ ; \quad f_k = K(x_1 - x)$$

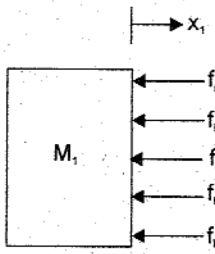


Fig 2 : Free body diagram of mass M_1 (node 1).

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1s^2X_1(s) + B_1sX_1(s) + Bs\,[X_1(s) - X(s)] + K_1X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore X_1(s) = X(s)\,\frac{Bs + K}{M_1s^2 + (B_1 + B)\,s + (K_1 + K)} \qquad \text{......(1)}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are marked as f_{m2}, f_{b2}, f_b and f_k.

$$f_{m2} = M_2\,\frac{d^2x}{dt^2} \quad ; \qquad f_{b2} = B_2\,\frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}\,(x - x_1) \quad ; \qquad f_k = K(x - x_1)$$

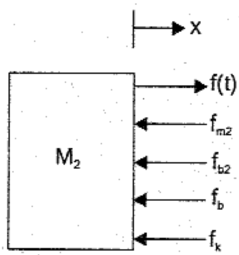


Fig 3 : Free body diagram of mass M_2 (node 2).

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2\,\frac{d^2x}{dt^2} + B_2\,\frac{dx}{dt} + B\frac{d}{dt}\,(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2s^2X(s) + B_2sX(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \qquad \text{.....(2)}$$

Substituting for $X_1(s)$ from equation (1) in equation (2) we get,

$$X(s)\,[M_2s^2 + (B_2 + B)s + K] - X(s)\,\frac{(Bs + K)^2}{M_1s^2 + (B_1 + B)s + (K_1 + K)} = F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K]\,[M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

RESULT

The differential equations governing the system are,

1. $$M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B\frac{d}{dt}(x_1-x)+K_1x_1+K(x_1-x)=0$$

2. $$M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d}{dt}(x-x_1)+K(x-x_1)=f(t)$$

The transfer function of the system is,

$$\frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)\,s+(K_1+K)}{\left[M_1s^2+(B_1+B)\,s+(K_1+K)\right]\left[M_2s^2+(B_2+B)\,s+K\right]-(Bs+K)^2}$$

EXAMPLE 1.2

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig 1.

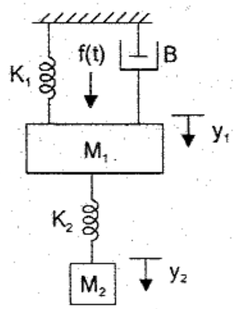


Fig 1.

SOLUTION

Let, Laplace transform of $f(t)=\mathcal{L}\{f(t)\}=F(s)$

Laplace transform of $y_1=\mathcal{L}\{y_1\}=Y_1(s)$

Laplace transform of $y_2=\mathcal{L}\{y_2\}=Y_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes.

The free body diagram of mass M_1 is shown in fig 2.

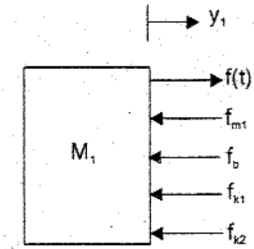


Fig 2.

The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1}=M_1\frac{d^2y_1}{dt^2}\ ;\quad f_b=B\frac{dy_1}{dt}\ ;\quad f_{k1}=K_1y_1\ ;\quad f_{k2}=K_2(y_1-y_2)$$

By Newton's second law, $f_{m1}+f_b+f_{k1}+f_{k2}=f(t)$

$$\therefore M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t)\qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial condition we get,

$$M_1s^2Y_1(s)+BsY_1(s)+K_1Y_1(s)+K_2[Y_1(s)-Y_2(s)]=F(s)$$

$$Y_1(s)[M_1s^2+Bs+(K_1+K_2)]-Y_2(s)K_2=F(s)\qquad .....(2)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on M_2 are f_{m2} and f_{k2}.

$$f_{m2}=M_2\frac{d^2y_2}{dt^2}\ ;\qquad f_{k2}=K_2(y_2-y_1)$$

By Newton's second law, $f_{m2}+f_{k2}=0$

$$\therefore M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$$

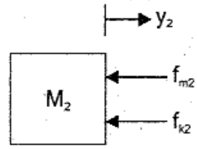


Fig 3.

On taking Laplace transform of above equation we get,

$$M_2s^2Y_2(s)+K_2[Y_2(s)-Y_1(s)]=0$$

$$Y_2(s)\;[M_2s^2+K_2]-Y_1(s)\;K_2=0$$

$$\therefore Y_1(s)=Y_2(s)\;\frac{M_2s^2+K_2}{K_2} \qquad .....(3)$$

Substituting for $Y_1(s)$ from equation (3) in equation (2) we get,

$$Y_2(s)\left[\frac{M_2s^2+K_2}{K_2}\right]\left[M_1s^2+Bs+(K_1+K_2)\right]-Y_2(s)\,K_2=F(s)$$

$$Y_2(s)\left[\frac{(M_2s^2+K_2)\;[M_1s^2+Bs+(K_1+K_2)]-K_2^2}{K_2}\right]=F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+(K_1+K_2)\right]\left[M_2s^2+K_2\right]-K_2^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\dfrac{d^2y_1}{dt^2}+B\dfrac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t)$

2. $M_2\dfrac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$

The transfer function of the system is,

$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+\;(K_1+K_2)\right]\left[M_2s^2+K_2\right]-K_2^2}$$

EXAMPLE 1.3

Determine the transfer function, $\dfrac{X_1(s)}{F(s)}$ and $\dfrac{X_2(s)}{F(s)}$ for the system shown in fig 1.

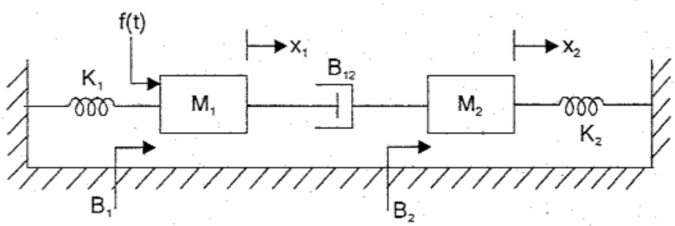


Fig 1.

SOLUTION

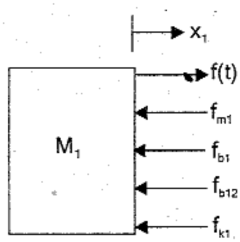


Fig 2.

Let, Laplace transform of f(t) = $\mathcal{L}\{f(t)\}$ = F(s)

Laplace transform of x_1 = $\mathcal{L}\{x_1\}$ = $X_1(s)$

Laplace transform of x_2 = $\mathcal{L}\{x_2\}$ = $X_2(s)$

The system has two nodes and they are mass M_1 and M_2. The differential equations governing the system are the force balance equations at these nodes. The free body diagram of mass M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \;;\; f_{b1} = B_1 \frac{dx_1}{dt} \;;\; f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \;;\; f_{k1} = K_1 x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d(x_1 - x_2)}{dt} + K_1 x_1 = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1 s^2 X_1(s) + B_1 s\, X_1(s) + B_{12} s\, [X_1(s) - X_2(s)] + K_1 X_1(s) = F(s)$$

$$X_1(s)\, [M_1 s^2 + (B_1 + B_{12})\, s + K_1] - B_{12} s X_2(s) = F(s) \quad .....(1)$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b12} and f_{k2}.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1) \;; \quad f_{k2} = K_2 x_2$$

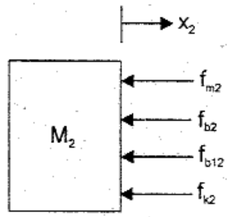


Fig 3.

By Newton's second law, $f_{m2} + f_{b2} + f_{b12} + f_{k2} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d(x_2 - x_1)}{dt} + K_2 x_2 = 0 \quad .....(2)$$

On taking Laplace transform of equation (2) with zero initial conditions we get,

$$M_2 s^2 X_2(s) + B_2 s X_2(s) + B_{12} s\, [X_2(s) - X_1(s)] + K_2 X_2(s) = 0$$

$$X_2(s)\, [M_2 s^2 + (B_2 + B_{12})\, s + K_2] - B_{12}\, s\, X_1(s) = 0$$

$$X_2(s)\, [M_2 s^2 + (B_2 + B_{12})\, s + K_2] = B_{12}\, s\, X_1(s)$$

$$X_2(s) = \frac{B_{12} s\, X_1(s)}{[M_2 s^2 + (B_2 + B_{12})\, s + K_2]} \quad .....(3)$$

Substituting for $X_2(s)$ from equation (3) in equation (1) we get,

$$X_1(s)\, [M_1 s^2 + (B_1 + B_{12})\, s + K_1] - \frac{(B_{12} s)^2\, X_1(s)}{M_2 s^2 + (B_2 + B_{12})\, s + K_2} = F(s)$$

$$\frac{X_1(s)\left[[M_1 s^2 + (B_1 + B_{12})\, s + K_1]\,[M_2 s^2 + (B_2 + B_{12})\, s + K_2] - (B_{12} s)^2\right]}{M_2 s^2 + (B_2 + B_{12})\, s + K_2} = F(s)$$

$$\therefore \frac{X_1(s)}{F(s)} = \frac{M_2 s^2 + (B_2 + B_{12})\, s + K_2}{[M_1 s^2 + (B_1 + B_{12})\, s + K_1]\,[M_2 s^2 + (B_2 + B_{12})\, s + K_2] - (B_{12} s)^2}$$

From equation (3) we get,

$$X_1(s) = \frac{[M_2 s^2 + (B_2 + B_{12})\, s + K_2]\, X_2(s)}{B_{12} s} \quad .....(4)$$

Substituting for $X_1(s)$ from equation (4) in equation (1) we get,

$$\frac{X_2(s)\, [M_2 s^2 + (B_2 + B_{12})\, s + K_2]}{B_{12} s}\, [M_1 s^2 + (B_1 + B_{12})\, s + K_1] - B_{12} s\, X_2(s) = F(s)$$

$$X_2(s)\left[\frac{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}{B_{12}s}\right]=F(s)$$

$$\therefore \frac{X_2(s)}{F(s)}=\frac{B_{12}s}{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}$$

RESULT

The differential equations governing the system are,

1. $M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d(x_1-x_2)}{dt}+K_1x_1=f(t)$
2. $M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+B_{12}\frac{d(x_2-x_1)}{dt}+K_2x_2=0$

The transfer functions of the system are,

1. $\frac{X_1(s)}{F(s)}=\frac{M_2s^2+(B_2+B_{12})\,s+K_2}{[M_1s^2+(B_1+B_{12})\,s+K_1]\,[M_2s^2+(B_2+B_{12})\,s+K_2]-(B_{12}s)^2}$
2. $\frac{X_2(s)}{F(s)}=\frac{B_{12}s}{[M_2s^2+(B_2+B_{12})\,s+K_2]\,[M_1s^2+(B_1+B_{12})\,s+K_1]-(B_{12}s)^2}$

EXAMPLE 1.4

Write the equations of motion in s-domain for the system shown in fig 1. Determine the transfer function of the system.

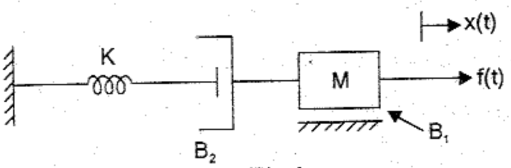


Fig 1.

SOLUTION

Let, Laplace transform of $x(t)=\mathcal{L}\{x(t)\}=X(s)$

Laplace transform of $f(t)=\mathcal{L}\{f(t)\}=F(s)$

Let x_1 be the displacement at the meeting point of spring and dashpot. Laplace transform of x_1 is $X_1(s)$.

The system has two nodes and they are mass M and the meeting point of spring and dashpot. The differential equations governing the system are the force balance equations at these nodes. The equations of motion in the s-domain are obtained by taking Laplace transform of the differential equations.

The free body diagram of mass M is shown in fig 2. The opposing forces are marked as f_m, f_{b1} and f_{b2}.

$$f_m=M\frac{d^2x}{dt^2}\quad;\quad f_{b1}=B_1\frac{dx}{dt}\quad;\quad f_{b2}=B_2\frac{d}{dt}(x-x_1)$$

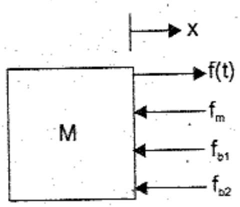


Fig 2.

By Newton's second law the force balance equation is,

$$f_m+f_{b1}+f_{b2}=f(t)$$

$$\therefore M\frac{d^2x}{dt^2}+B_1\frac{dx}{dt}+B_2\frac{d}{dt}(x-x_1)=f(t)$$

On taking Laplace transform of the above equation we get,

$$Ms^2\,X(s)+B_1\,s\,X(s)+B_2\,s\,[X(s)-X_1(s)]=F(s)$$

$$\left[Ms^2+(B_1+B_2)\,s\right]X(s)-B_2\,s\,X_1(s)=F(s) \qquad .....(1)$$

The free body diagram at the meeting point of spring and dashpot is shown in fig 3. The opposing forces are marked as f_k and f_{b2}.

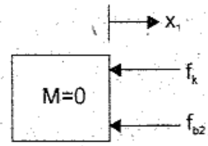


Fig 3.

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x)\,; \quad f_k = K\,x_1$$

By Newton's second law, $f_{b2} + f_k = 0$

$$\therefore B_2 \frac{d}{dt}(x_1 - x) + K\,x_1 = 0$$

On taking Laplace transform of the above equation we get,

$$B_2\,s\,[X_1(s) - X(s)] + K\,X_1(s) = 0$$

$$(B_2\,s + K)\,X_1(s) - B_2\,s\,X(s) = 0$$

$$\therefore X_1(s) = \frac{B_2\,s}{B_2\,s + K}\,X(s) \qquad .....(2)$$

Substituting for $X_1(s)$ from equation (2) in equation (1) we get,

$$\left[M\,s^2 + (B_1 + B_2)\,s\right] X(s) - B_2\,s\left[\frac{B_2\,s}{B_2\,s + K}\right] X(s) = F(s)$$

$$X(s)\,\frac{\left[[M\,s^2 + (B_1 + B_2)\,s]\,(B_2\,s + K) - (B_2\,s)^2\right]}{B_2\,s + K} = F(s)$$

$$\therefore \frac{X(s)}{F(s)} = \frac{B_2\,s + K}{[M\,s^2 + (B_1 + B_2)\,s]\,(B_2\,s + K) - (B_2\,s)^2}$$

RESULT

The differential equations governing the system are,

1. $M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d}{dt}(x - x_1) = f(t)$
2. $B_2\frac{d}{dt}(x_1 - x) + K\,x_1 = 0$

The equations of motion in s-domain are,

1. $[M\,s^2 + (B_1 + B_2)\,s]\,X(s) - B_2\,s\,X_1(s) = F(s)$
2. $(B_2\,s + K)\,X_1(s) - B_2\,s\,X(s) = 0$

The transfer function of the system is,

$$\frac{X(s)}{F(s)} = \frac{B_2\,s + K}{[M\,s^2 + (B_1 + B_2)\,s]\,(B_2\,s + K) - (B_2\,s)^2}$$

1.5 MECHANICAL ROTATIONAL SYSTEMS

The model of rotational mechanical systems can be obtained by using three elements, ***moment of inertia*** [J] of mass, ***dash-pot*** with rotational frictional coefficient [B] and ***torsional spring*** with stiffness [K].

The weight of the rotational mechanical system is represented by the moment of inertia of the mass. The moment of inertia of the system or body is considered to be concentrated at the centre of gravity of the body. The elastic deformation of the body can be represented by a spring (torsional spring). The friction existing in rotational mechanical system can be represented by the dash-pot. The dash-pot is a piston rotating inside a cylinder filled with viscous fluid.

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system. The torques acting on a rotational mechanical body are governed by ***Newton's second law of motion*** for rotational systems. It states that the sum of torques acting on a body is zero (or Newton's law states that the sum of applied torques is equal to the sum of opposing torques on a body).

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

TORQUE BALANCE EQUATIONS OF IDEALISED ELEMENTS

Consider an ideal mass element shown in fig 1.14 which has negligible friction and elasticity. The opposing torque due to moment of inertia is proportional to the angular acceleration.

Let, T = Applied torque.

T_j = Opposing torque due to moment of inertia of the body.

Here $T_j \propto \frac{d^2\theta}{dt^2}$ or $T_j = J\frac{d^2\theta}{dt^2}$

By Newton's second law,

$$\boxed{T = T_j = J\frac{d^2\theta}{dt^2}} \qquad .....(1.7)$$

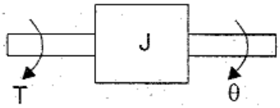


Fig 1.14 : Ideal rotational mass element.

Consider an ideal frictional element dash pot shown in fig 1.15 which has negligible moment of inertia and elasticity. Let a torque be applied on it. The dash pot will offer an opposing torque which is proportional to the angular velocity of the body.

Let, T = Applied torque.

T_b = Opposing torque due to friction.

$T_b \propto \frac{d\theta}{dt}$ or $T_b = B\frac{d\theta}{dt}$

By Newton's second law, $\boxed{T = T_b = B\frac{d\theta}{dt}}$ .....(1.8)

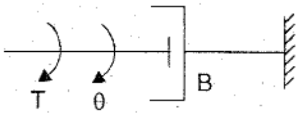


Fig 1.15 : Ideal rotational dash-pot with one end fixed to reference.

When the dash pot has angular displacement at both ends as shown in fig 1.16, the opposing torque is proportional to the differential angular velocity.

$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$ or $T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)} \qquad .....(1.9)$$

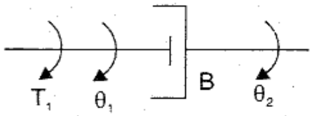


Fig 1.16 : Ideal dash-pot with angular displacement at both ends.

Consider an ideal elastic element, torsional spring as shown in fig 1.17, which has negligible moment of inertia and friction. Let a torque be applied on it. The torsional spring will offer an opposing torque which is proportional to angular displacement of the body.

Let, T = Applied torque.

T_k = Opposing torque due to elasticity.

$T_k \propto \theta$ or $T_k = K\theta$

By Newton's second law, $\boxed{T = T_k = K\theta}$ .....(1.10)

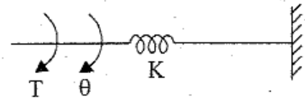


Fig 1.17 : Ideal spring with one end fixed to reference.

When the spring has angular displacement at both ends as shown in fig 1.18 the opposing torque is proportional to differential angular displacement.

$T_k \propto (\theta_1 - \theta_2)$ or $T_k = K(\theta_1 - \theta_2)$

$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$ .....(1.11)

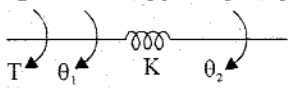


Fig 1.18 : Ideal spring with angular displacement at both ends.

Guidelines to determine the Transfer Function of Mechanical Rotational System

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element.

2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ_1, θ_2, θ_3, etc., and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque.

4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.

5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.

6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system.

> ***Note :***
>
> *Laplace transform of* $\theta = \mathcal{L}\{\theta\} = \theta(s)$
>
> *Laplace transform of* $\dfrac{d\theta}{dt} = \mathcal{L}\left\{\dfrac{d\theta}{dt}\right\} = s\,\theta(s)$ *(with zero initial conditions)*
>
> *Laplace transform of* $\dfrac{d^2\theta}{dt^2} = \mathcal{L}\left\{\dfrac{d^2\theta}{dt^2}\right\} = s^2\,\theta(s)$ *(with zero initial conditions)*

EXAMPLE 1.5

Write the differential equations governing the mechanical rotational system shown in fig 1. Obtain the transfer function of the system.

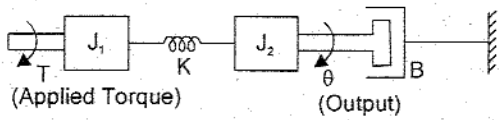


Fig 1.

SOLUTION

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of T $= \mathcal{L}\{T\} = T(s)$

Laplace transform of θ $= \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad .....(1)$$

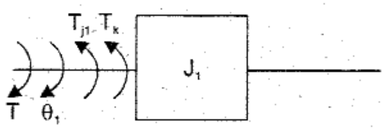


Fig 2 : Free body diagram of mass with moment of inertia J_1.

On taking Laplace transform of equation (1) with zero initial conditions we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\, \theta_1(s) - K\, \theta(s) = T(s) \qquad .....(2)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques acting on J_2 are marked as T_{j2}, T_b and T_k.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

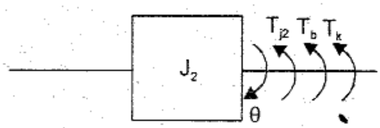


Fig 3 : Free body diagram of mass with moment of inertia J_2.

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2 s^2\theta(s) + B\, s\, \theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \quad .....(3)$$

Substituting for $\theta_1(s)$ from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are,

1. $J_1 \dfrac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$

2. $J_2 \dfrac{d^2\theta}{dt^2} + B\dfrac{d\theta}{dt} + K\theta - K\theta_1 = 0$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

EXAMPLE 1.6

Write the differential equations governing the mechanical rotational system shown in fig 1. and determine the transfer function $\theta(s)/T(s)$.

SOLUTION

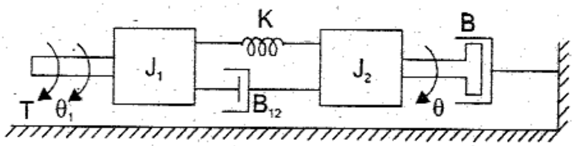


Fig 1.

In the given system, the torque T is the input and the angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

The system has two nodes and they are masses with moment of inertia J_1 and J_2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J_1 be θ_1. The free body diagram of J_1 is shown in fig 2. The opposing torques acting on J_1 are marked as T_{j1}, T_{b12} and T_k.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

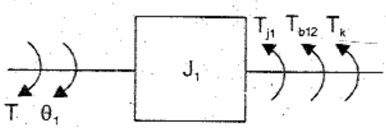


Fig 2 : Free body diagram of mass with moment of inertia J_1.

By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1\frac{d^2\theta_1}{dt^2}+B_{12}\frac{d}{dt}(\theta_1-\theta)+K(\theta_1-\theta)=T$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_1s^2\theta_1(s)+s\,B_{12}\,[\theta_1(s)-\theta(s)]+K\theta_1(s)-K\theta(s)=T(s)$$

$$\theta_1(s)\,[J_1s^2+sB_{12}+K]-\theta(s)\,[sB_{12}+K]=T(s) \qquad .....(1)$$

The free body diagram of mass with moment of inertia J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b12}, T_b and T_k.

$$T_{j2}=J_2\frac{d^2\theta}{dt^2}\quad ; \quad T_{b12}=B_{12}\frac{d}{dt}(\theta-\theta_1)$$

$$T_b=B\frac{d\theta}{dt}\quad ; \quad T_k=K(\theta-\theta_1)$$

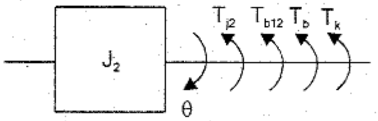


Fig 3 : Free body diagram of mass with moment of inertia J_2.

By Newton's second law, $T_{j2}+T_{b12}+T_b+T_k=0$

$$J_2\frac{d^2\theta}{dt^2}+B_{12}\frac{d}{dt}(\theta-\theta_1)+B\frac{d\theta}{dt}+K(\theta-\theta_1)=0$$

$$J_2\frac{d^2\theta}{dt^2}-B_{12}\frac{d\theta_1}{dt}+\frac{d\theta}{dt}(B_{12}+B)+K\theta-K\theta_1=0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$J_2s^2\theta(s)-B_{12}s\theta_1(s)+s\theta(s)\,[B_{12}+B]+K\theta(s)-K\theta_1(s)=0$$

$$\theta(s)\,[s^2J_2+s(B_{12}+B)+K]-\theta_1(s)\,[sB_{12}+K]=0$$

$$\theta_1(s)=\frac{[s^2J_2+s(B_{12}+B)+K]}{[sB_{12}+K]}\,\theta(s) \qquad .....(2)$$

Substituting for $\theta_1(s)$ from equation (2) in equation (1) we get,

$$[J_1s^2+sB_{12}+K]\;\frac{[J_2s^2+s(B_{12}+B)+K]\,\theta(s)}{(sB_{12}+K)}-(sB_{12}+K)\,\theta(s)=T(s)$$

$$\left[\frac{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}{(sB_{12}+K)}\right]\theta(s)=T(s)$$

$$\therefore\frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

RESULT

The differential equations governing the system are,

1. $J_1\dfrac{d^2\theta_1}{dt^2}+B_{12}\dfrac{d}{dt}(\theta_1-\theta)+K(\theta_1-\theta)=T$

2. $J_2\dfrac{d^2\theta}{dt^2}-B_{12}\dfrac{d\theta_1}{dt}+\dfrac{d\theta}{dt}(B_{12}+B)+K(\theta-\theta_1)=0$

The transfer function of the system is,

$$\frac{\theta(s)}{T(s)}=\frac{(sB_{12}+K)}{(J_1s^2+sB_{12}+K)\,[J_2s^2+s(B_{12}+B)+K]-(sB_{12}+K)^2}$$

1.6 ELECTRICAL SYSTEMS

The models of electrical systems can be obtained by using resistor, capacitor and inductor. The current-voltage relation of resistor, inductor and capacitor are given in table-1. For modelling electrical systems, the electrical network or equivalent circuit is formed by using R, L and C and voltage or current source.

The differential equations governing the electrical systems can be formed by writing Kirchoff 's current law equations by choosing various nodes in the network or Kirchoff 's voltage law equations by choosing various closed paths in the network. The transfer function can be obtained by taking Laplace transform of the differential equations and rearranging them as a ratio of output to input.

TABLE-1.1 : Current-Voltage Relation of R, L and C

Element	Voltage across the element	Current through the element
i(t) R + − v(t)	$v(t) = Ri(t)$	$i(t) = \dfrac{v(t)}{R}$
i(t) L + − v(t)	$v(t) = L\dfrac{d}{dt}i(t)$	$i(t) = \dfrac{1}{L}\int v(t)\,dt$
i(t) C + − v(t)	$v(t) = \dfrac{1}{C}\int i(t)\,dt$	$i(t) = C\dfrac{dv(t)}{dt}$

EXAMPLE 1.7

Obtain the transfer function of the electrical network shown in fig 1.

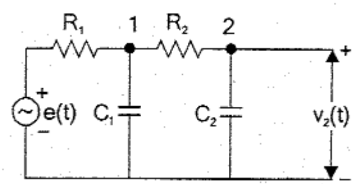


Fig 1.

SOLUTION

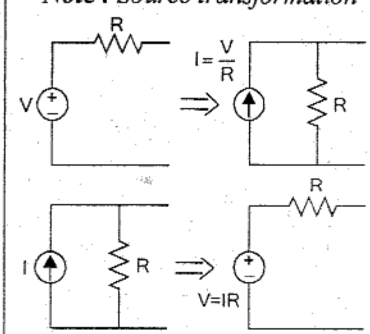
In the given network, input is e(t) and output is $v_2(t)$.

Let, Laplace transform of $e(t) = \mathcal{L}\{e(t)\} = E(s)$

Laplace transform of $v_2(t) = \mathcal{L}\{v_2(t)\} = V_2(s)$

The transfer function of the network is $\dfrac{V_2(s)}{E(s)}$

Note : Source transformation



Transform the voltage source in series with resistance R_1 into equivalent current source as shown in figure 2. The network has two nodes. Let the node voltages be v_1 and v_2. The Laplace transform of node voltages v_1 and v_2 are $V_1(s)$ and $V_2(s)$ respectively. The differential equations governing the network are given by the Kirchoff 's current law equations at these nodes.

At node-1, by Kirchoff's current law (refer fig 3)

$$\frac{v_1}{R_1} + C_1\frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$$

On taking Laplace transform of above equation with zero initial conditions we get,

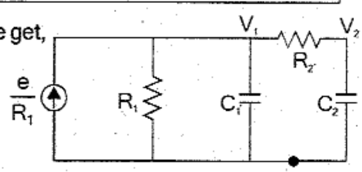


Fig 2.

$$\frac{V_1(s)}{R_1} + C_1 sV_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \qquad .....(1)$$

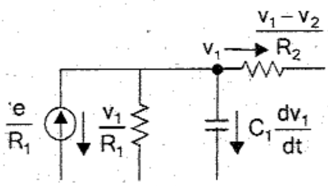


Fig 3.

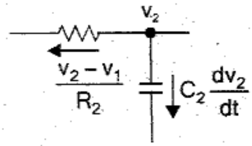


Fig 4.

At node-2, by Kirchoff's current law (refer fig 4)

$$\frac{v_2 - v_1}{R_2} + C_2 \frac{dv_2}{dt} = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2 s\, V_2(s) = 0$$

$$\frac{V_1(s)}{R_2} = \frac{V_2(s)}{R_2} + C_2 s\, V_2(s) = \left[\frac{1}{R_2} + sC_2\right] V_2(s)$$

$$\therefore V_1(s) = \left[1 + sC_2R_2\right] V_2(s) \qquad .....(2$$

Substituting for $V_1(s)$ from equation (2) in equation (1) we get,

$$(1 + sR_2C_2)\, V_2(s) \left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$\left[\frac{(1 + sR_2C_2)\,(R_2 + R_1 + sC_1R_1R_2) - R_1}{R_1R_2}\right] V_2(s) = \frac{E(s)}{R_1}$$

$$\therefore \frac{V_2(s)}{E(s)} = \frac{R_2}{\left[(1 + sR_2C_2)\,(R_1 + R_2 + sC_1R_1R_2) - R_1\right]}$$

RESULT

The (node basis) differential equations governing the electrical network are,

1. $\frac{v_1}{R_1} + C_1 \frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} = \frac{e}{R_1}$

2. $\frac{v_2 - v_1}{R_2} + C_2 \frac{dv_2}{dt} = 0$

The transfer function of the electrical network is,

$$\frac{V_2(s)}{E(s)} = \frac{R_2}{\left[(1 + sR_2C_2)\,(R_1 + R_2 + sC_1R_1R_2) - R_1\right]}$$

1.7 TRANSFER FUNCTION OF ARMATURE CONTROLLED DC MOTOR

The speed of DC motor is directly proportional to armature voltage and inversely proportional flux in field winding. In armature controlled DC motor the desired speed is obtained by varying t armature voltage. This speed control system is an electro-mechanical control system. The electric system consists of the armature and the field circuit but for analysis purpose, only the armature circuit considered because the field is excited by a constant voltage. The mechanical system consists of t rotating part of the motor and load connected to the shaft of the motor. The armature controlled I motor speed control system is shown in fig 1.19.

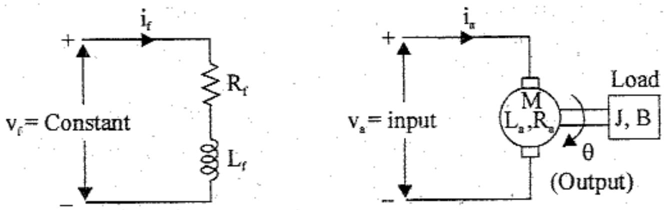


Fig 1.19 : Armature controlled DC motor.

Let, R_a = Armature resistance, Ω
L_a = Armature inductance, H
i_a = Armature current, A
v_a = Armature voltage, V
e_b = Back emf, V
K_t = Torque constant, N-m/A
T = Torque developed by motor, N-m
θ = Angular displacement of shaft, rad
J = Moment of inertia of motor and load, Kg-m²/rad
B = Frictional coefficient of motor and load, N-m/(rad/sec)
K_b = Back emf constant, V/(rad/sec)

The equivalent circuit of armature is shown in fig 1.20.

By Kirchoff's voltage law, we can write,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \qquad .....(1.12)$$

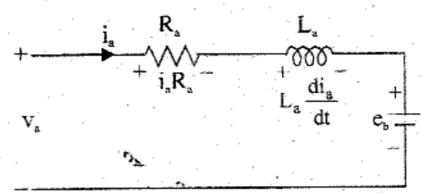


Fig 1.20 : Equivalent circuit of armature.

Torque of DC motor is proportional to the product of flux and current. Since flux is constant in this system, the torque is proportional to i_a alone.

$$T \propto i_a$$

$$\therefore \text{Torque, } T = K_t i_a \qquad .....(1.13)$$

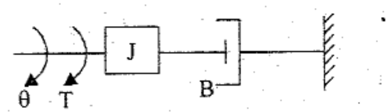


Fig 1.21.

The mechanical system of the motor is shown in fig 1.21.

The differential equation governing the mechanical system of motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.14)$$

The back emf of DC machine is proportional to speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \quad \text{or} \quad \text{Back emf, } e_b = K_b \frac{d\theta}{dt} \qquad .....(1.15)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$$\mathcal{L}\{v_a\} = V_a(s); \quad \mathcal{L}\{e_b\} = E_b(s); \quad \mathcal{L}\{T\} = T(s); \quad \mathcal{L}\{i_a\} = I_a(s); \quad \mathcal{L}\{\theta\} = \theta(s)$$

The differential equations governing the armature controlled DC motor speed control system are,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = v_a \quad ; \quad T = K_t i_a \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad ; \quad e_b = K_b \frac{d\theta}{dt}$$

Taking Laplace transform of the above equations with zero initial conditions we get,

$$I_a(s)\,R_a + L_a s I_a(s) + E_b(s) = V_a(s) \qquad .....(1.16)$$

$$T(s) = K_t I_a(s) \qquad .....(1.17)$$

$$Js^2\theta(s) + B\,s\,\theta(s) = T(s) \qquad .....(1.18)$$

$$E_b(s) = K_b s\,\theta(s) \qquad .....(1.19)$$

On equating equations (1.17) and (1.18) we get,

$$K_t I_a(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_a(s) = \frac{(Js^2 + Bs)}{K_t}\,\theta(s) \qquad .....(1.20)$$

Equation (1.16) can be written as,

$$(R_a + sL_a)\,I_a(s) + E_b(s) = V_a(s) \qquad .....(1.21)$$

Substituting for $E_b(s)$ and $I_a(s)$ from equation (1.19) and (1.20) respectively in equation (1.21),

$$(R_a + sL_a)\,\frac{(Js^2 + Bs)}{K_t}\theta(s) + K_b s\,\theta(s) = V_a(s)$$

$$\left[\frac{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s}{K_t}\right]\theta(s) = V_a(s)$$

The required transfer function is $\dfrac{\theta(s)}{V_a(s)}$

$$\therefore \frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s} \qquad .....(1.22)$$

$$= \frac{K_t}{R_a Js^2 + R_a Bs + L_a Js^3 + L_a Bs^2 + K_b K_t s}$$

$$= \frac{K_t}{s\left[JL_a s^2 + (JR_a + BL_a)\,s + (BR_a + K_b K_t)\right]}$$

$$= \frac{K_t / JL_a}{s\left[s^2 + \left(\dfrac{JR_a + BL_a}{JL_a}\right)s + \left(\dfrac{BR_a + K_b K_t}{JL_a}\right)\right]} \qquad .....(1.23)$$

The transfer function of armature controlled dc motor can be expressed in another standard form as shown below. From equation (1.22) we get,

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{(R_a + sL_a)\,(Js^2 + Bs) + K_b K_t s} = \frac{K_t}{R_a\left(\dfrac{sL_a}{R_a} + 1\right) Bs\left(1 + \dfrac{Js^2}{Bs}\right) + K_b K_t\,s}$$

$$= \frac{K_t / R_a B}{s\left[(1 + sT_a)\,(1 + sT_m) + \dfrac{K_b K_t}{R_a B}\right]} \qquad .....(1.24)$$

where, $\frac{L_a}{R_a} = T_a$ = Electrical time constant

$\frac{J}{B} = T_m$ = Mechanical time constant

1.8 TRANSFER FUNCTION OF FIELD CONTROLLED DC MOTOR

The speed of a DC motor is directly proportional to armature voltage and inversely proportional to flux. In field controlled DC motor the armature voltage is kept constant and the speed is varied by varying the flux of the machine. Since flux is directly proportional to field current, the flux is varied by varying field current. The speed control system is an electromechanical control system. The electrical system consists of armature and field circuit but for analysis purpose, only field circuit is considered because the armature is excited by a constant voltage. The mechanical system consists of the rotating part of the motor and the load connected to the shaft of the motor. The field controlled DC motor speed control system is shown in fig 1.22.

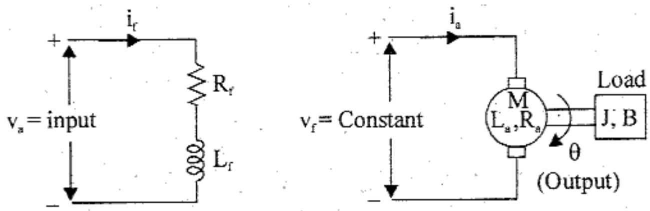


Fig 1.22 : Field controlled DC motor.

Let, R_f = Field resistance, Ω

L_f = Field inductance, H

i_f = Field current, A

v_f = Field voltage, V

T = Torque developed by motor, N-m

K_{tf} = Torque constant, N-m/A

J = Moment of inertia of rotor and load, Kg-m²/rad

B = Frictional coefficient of rotor and load, N-m/(rad/sec)

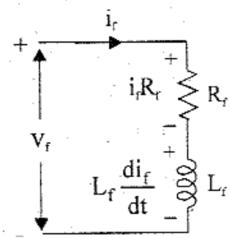


Fig 1.23 : Equivalent circuit of field.

The equivalent circuit of field is shown in fig 1.23.

By Kirchoff's voltage law, we can write

$$R_f i_f + L_f \frac{di_f}{dt} = v_f \qquad .....(1.25)$$

The torque of DC motor is proportional to product of flux and armature current. Since armature current is constant in this system, the torque is proportional to flux alone, but flux is proportional to field current.

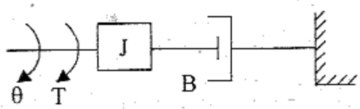


Fig 1.24.

$$T \propto i_f, \quad \therefore \text{Torque, } T = K_{tf}\, i_f \qquad .....(1.26)$$

The mechanical system of the motor is shown in fig 1.24. The differential equation governing the mechanical system of the motor is given by,

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \qquad .....(1.27)$$

The Laplace transform of various time domain signals involved in this system are shown below.

$\mathcal{L}\{i_f\} = I_f(s)$; $\mathcal{L}\{T\} = T(s)$; $\mathcal{L}\{v_f\} = V_f(s)$; $\mathcal{L}\{\theta\} = \theta(s)$

The differential equations governing the field controlled DC motor are,

$$K_f i_f + L_f \frac{di_f}{dt} = v_f \quad ; \quad T = K_{tf} i_f \quad ; \quad J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T$$

On taking Laplace transform of the above equations with zero initial condition we get,

$$R_f I_f(s) + L_f s\, I_f(s) = V_f(s) \qquad .....(1.28)$$

$$T(s) = K_{tf}\, I_f(s) \qquad .....(1.29)$$

$$Js^2\theta(s) + Bs\theta(s) = T(s) \qquad .....(1.30)$$

Equating equations (1.29) and (1.30) we get,

$$K_{tf}\, I_f(s) = Js^2\theta(s) + Bs\theta(s)$$

$$I_f(s) = s\,\frac{(Js+B)}{K_{tf}}\theta(s) \qquad .....(1.31)$$

The equation (1.28) can be written as,

$$(R_f + sL_f)\, I_f(s) = V_f(s) \qquad .....(1.32)$$

On substituting for $I_f(s)$ from equation (1.31) in equation (1.32) we get,

$$(R_f + sL_f)s\,\frac{(Js+B)}{K_{tf}}\,\theta(s) = V_f(s)$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s\,(R_f + sL_f)\,(B + sJ)}$$

$$= \frac{K_{tf}}{sR_f\left(1 + \frac{sL_f}{R_f}\right)B\left(1 + \frac{sJ}{B}\right)} = \frac{K_m}{s\,(1 + sT_f)\,(1 + sT_m)} \qquad .....(1.33)$$

where, $K_m = \frac{K_{tf}}{R_f B}$ = Motor gain constant

$T_f = \frac{L_f}{R_f}$ = Field time constant

$T_m = \frac{J}{B}$ = Mechanical time constant

1.9 ELECTRICAL ANALOGOUS OF MECHANICAL TRANSLATIONAL SYSTEMS

Systems remain ***analogous*** as long as the differential equations governing the systems or transfer functions are in identical form. The electric analogue of any other kind of system is of greater importance since it is easier to construct electrical models and analyse them.

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational systems are analogous to resistance, inductance and capacitance of electrical systems.

The input force in mechanical system is analogous to either voltage source or current source in electrical systems. The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies : ***force-voltage analogy*** and ***force-current analogy***.

FORCE-VOLTAGE ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-voltage analogy are shown in table-1.2.The table-1.3 shows the list of analogous quantities in force-voltage analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same velocity are said to be in series.
2. The elements having same velocity in mechanical system should have the same analogous current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of velocities of nodes (masses) in mechanical system.

TABLE- 1.2 : Analogous Elements in Force-Voltage Analogy

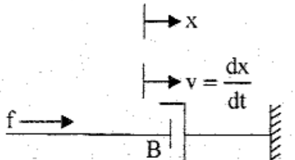
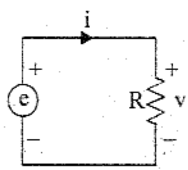
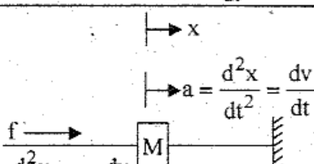
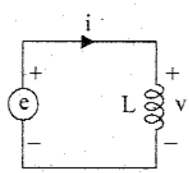
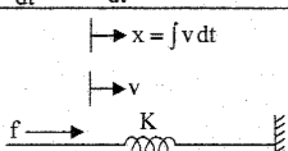
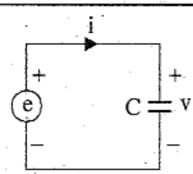
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $f = B\frac{dx}{dt} = Bv$	 $e = v$ and $v = Ri$ $\therefore e = Ri$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $f = Kx = K\int v\,dt$	 $e = v$ and $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

TABLE -1.3 : Analogous Quantities in Force-Voltage Analogy

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff's voltage law $\sum v = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

TABLE-1.4 : Analogous Elements in Force-Current Analogy

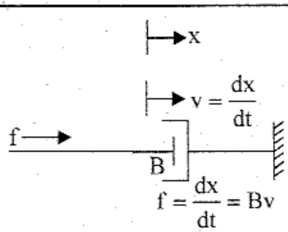
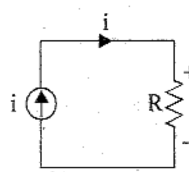
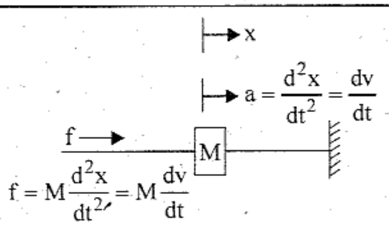
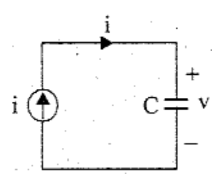
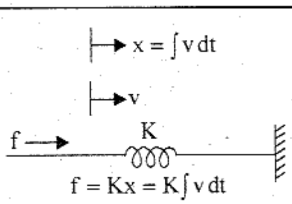
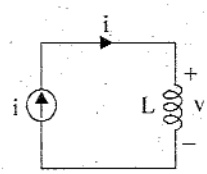
Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element
 $v = \frac{dx}{dt}$, $f = B\frac{dx}{dt} = Bv$	 $i = \frac{1}{R}v$
 $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$, $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $i = C\frac{dv}{dt}$
 $x = \int v\,dt$, $f = Kx = K\int v\,dt$	 $i = \frac{1}{L}\int v\,dt$

TABLE-1.5 : Analogous Quantities in Force-Current Analogy

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma f = 0$	Kirchoff's current law $\Sigma i = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

5. The mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.
6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

FORCE-CURRENT ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-current analogy are shown in table-1.4. The table-1.5 shows the list of analogous quantities in force-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-current analogy.

1. In electrical systems elements in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.
2. The elements having same velocity in mechanical system should have the same analogous voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. A mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (nodes) masses in mechanical system.
5. The mechanical driving sources (forces) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements connected to a node in electrical system.
6. The element connected between two nodes (masses) in mechanical system is represented as a common element between two nodes in electrical analogous system.

EXAMPLE 1.8

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

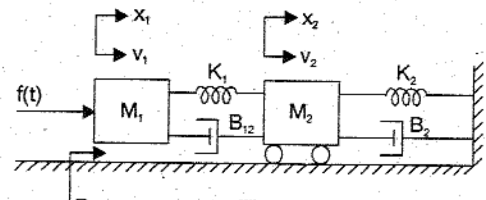


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{b12} and f_{k1}.

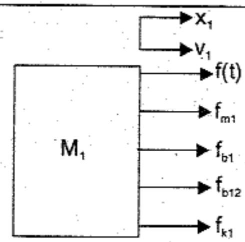


Fig 2.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt}$$

$$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2) \quad ; \quad f_{k1} = K_1(x_1 - x_2)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \qquad .....(1)$$

The free body diagram of M_2 is shown in fig 3.The opposing forces are marked as f_{m2}, f_{b2}, f_{b12}, f_{k1} and f_{k2}.

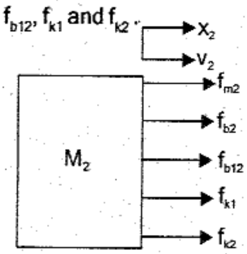


Fig 3.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx_2}{dt} \quad ; \quad f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$$

$$f_{k1} = K_1(x_2 - x_1) \quad ; \quad f_{k2} = K_2 x_2$$

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \qquad .....(2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) of the mechanical system we get,

$$\left(\text{i.e, } \frac{d^2x}{dt^2} = \frac{dv}{dt} \quad ; \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1 \int (v_1 - v_2)\,dt = f(t) \qquad .....(3)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_2 \int v_2\,dt + B_{12}(v_2 - v_1) + K_1 \int (v_2 - v_1)\,dt = 0 \qquad .....(4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit will have two meshes.

The force applied to mass, M_1 is represented by a voltage source in first mesh. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous element as common elements between two meshes. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow e(t)$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$v_1 \rightarrow i_1$	$M_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_2 \rightarrow 1/C_2$
$v_2 \rightarrow i_2$		$B_{12} \rightarrow R_{12}$	

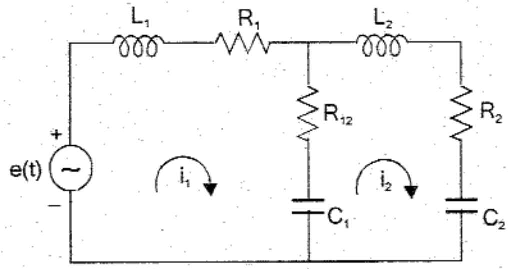


Fig 4 : Force-voltage electrical analogous circuit.

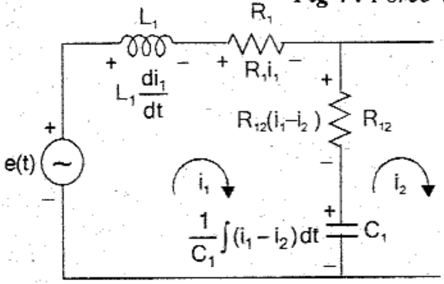


Fig 5.

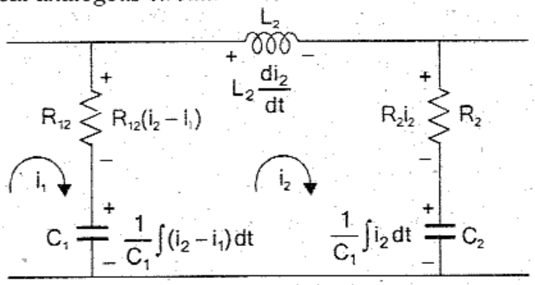


Fig 6.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt}+R_1i_1+R_{12}(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \quad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2\,dt+R_{12}(i_2-i_1)+\frac{1}{C_1}\int(i_2-i_1)dt=0 \quad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The elements M_1, B_1, K_1 and B_{12} are connected to first node. Hence they are represented by analogous elements connected to node-1 in analogous electrical circuit. The elements K_1, B_{12}, M_2, K_2, and B_2 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The elements K_1 and B_{12} are common between node-1 and 2 and so they are represented by analogous elements as common element between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$v_1 \rightarrow v_1$	$M_2 \rightarrow C_2$	$B_2 \rightarrow 1/R_2$	$K_2 \rightarrow 1/L_2$
$v_2 \rightarrow v_2$	$B_{12} \rightarrow 1/R_{12}$		

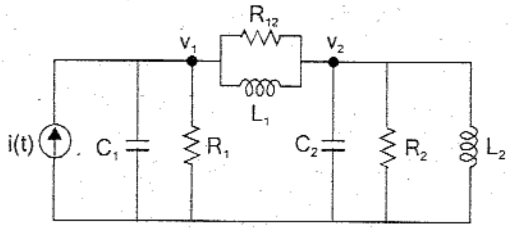


Fig 7 : Force-voltage electrical analogous circuit.

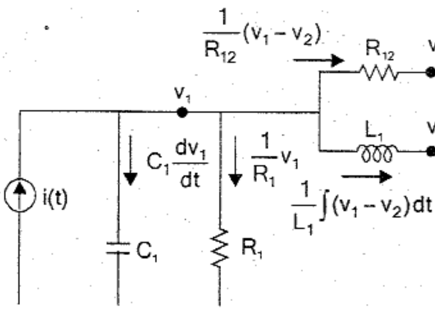


Fig 8.

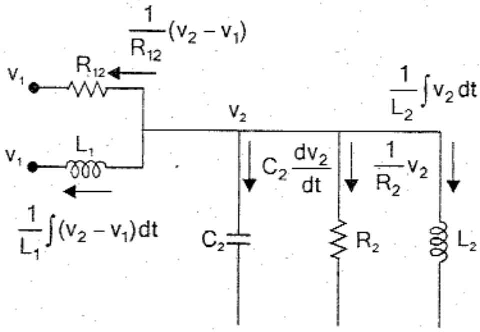


Fig 9.

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{R_{12}}(v_1 - v_2) + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t) \qquad .....(7)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{R_{12}}(v_2 - v_1) + \frac{1}{L_1}\int(v_2 - v_1)dt = 0 \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.9

Write the differential equations governing the mechanical system shown in fig 1. Draw the force -voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

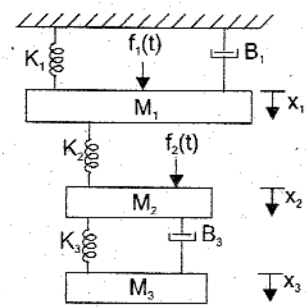


Fig 1.

SOLUTION

The given mechanical system has three nodes masses. The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1}, f_{k2} and f_{k1}.

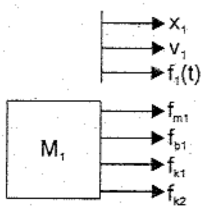


Fig 2.

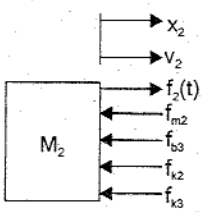


Fig 3.

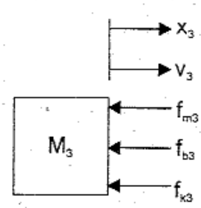


Fig 4.

$$f_{m1} = \frac{M_1 d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1\frac{dx_1}{dt} \quad ; \quad f_{k2} = K_2(x_1 - x_2) \quad ; \quad f_{k1} = K_1x_1$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k2} + f_{k1} = f_1(t)$

$$M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_2(x_1 - x_2) + K_1x_1 = f_1(t) \quad .....(1)$$

Free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b3}, f_{k2} & f_{k3}.

$$f_{m2} = M_2\frac{d^2x_2}{dt^2} \quad ; \quad f_{b3} = B_3\frac{d}{dt}(x_2 - x_3) \quad ; \quad f_{k2} = K_2(x_2 - x_1) \quad ; \quad f_{k3} = K_3(x_2 - x_3)$$

By Newton's second law, $f_{m2} + f_{b3} + f_{k2} + f_{k3} = f_2(t)$

$$M_2\frac{d^2x_2}{dt^2} + B_3\frac{d}{dt}(x_2 - x_3) + K_2(x_2 - x_1) + K_3(x_2 - x_3) = f_2(t) \quad .....(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3}, f_{b3} and f_{k3}.

$$f_{m3} = M_3\frac{d^2x_3}{dt^2} \quad ; \quad f_{b3} = B_3\frac{d}{dt}(x_3 - x_2) \quad ; \quad f_{k3} = K_3(x_3 - x_2)$$

By Newton's second law, $f_{m3} + f_{b3} + f_{k3} = 0$

$$M_3\frac{d^2x_3}{dt^2} + B_3\frac{d}{dt}(x_3 - x_2) + K_3(x_3 - x_2) = 0 \quad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt} \; ; \; \frac{dx}{dt} = v \text{ and } x = \int v\,dt\right)$$

$$M_1\frac{dv_1}{dt} + B_1v_1 + K_1\int v_1 dt + K_2\int(v_1 - v_2)dt = f_1(t) \quad .....(4)$$

$$M_2\frac{dv_2}{dt} + B_3(v_2 - v_3) + K_2\int(v_2 - v_1)dt + K_3\int(v_2 - v_3)dt = f_2(t) \quad .....(5)$$

$$M_3\frac{dv_3}{dt} + B_3(v_3 - v_2) + K_3\int(v_3 - v_2)dt = 0 \quad .....(6)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes. The force applied to mass, M_1 is represented by a voltage source in first mesh and the force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The elements M_3, K_3 and B_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh 1 and 2. The elements K_3 and B_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between mesh-2 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow e_1(t)$	$v_1 \rightarrow i_1$	$M_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
$f_2(t) \rightarrow e_2(t)$	$v_2 \rightarrow i_2$	$M_2 \rightarrow L_2$	$B_3 \rightarrow R_3$	$K_2 \rightarrow 1/C_2$
	$v_3 \rightarrow i_3$	$M_3 \rightarrow L_3$		$K_3 \rightarrow 1/C_3$

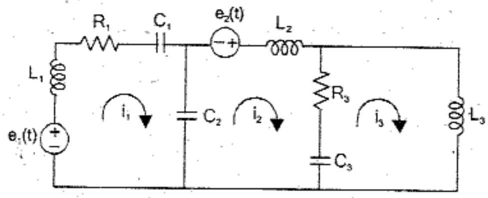


Fig 5 : Force-voltage electrical analogous circuit.

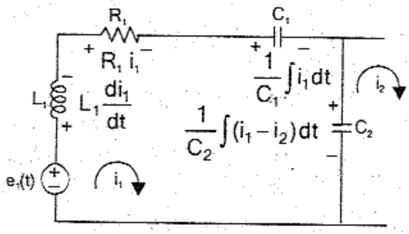


Fig 6.

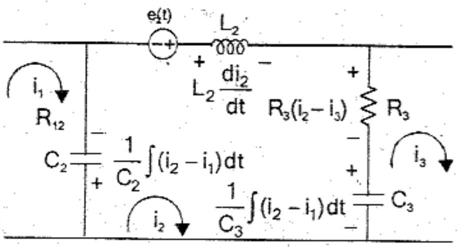


Fig 7.

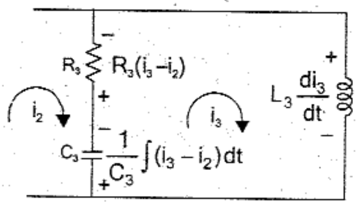


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7, 8).

$$L_1\frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + \frac{1}{C_2}\int (i_1 - i_2)dt = e_1(t) \quad .....(7)$$

$$L_2\frac{di_2}{dt} + R_3(i_2 - i_3) + \frac{1}{C_3}\int (i_2 - i_3)dt + \frac{1}{C_2}\int (i_2 - i_1)dt = e_2(t) \quad .....(8)$$

$$L_3\frac{di_3}{dt} + R_3(i_3 - i_2) + \frac{1}{C_3}\int (i_3 - i_2)dt = 0 \quad .....(9)$$

It is observed that the mesh equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) goveming the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will have three nodes.

The force applied to mass M_1 is represented as a current source connected to node-1 in analogous electrical circuit. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circuit.

The elements M_1, B_1, K_1 and K_2 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements M_2, B_3, K_2 and K_3 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements M_3, B_3 and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The element K_2 is common between node-1 and 2 and so it is represented by analogous element as common element between node-1 and 2 in analogous circuit. The elements B_3 and K_3 are common between node-2 and 3 and so they are represented by analogous elements as common elements between node-2 and 3. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$f_1(t) \rightarrow i_1(t)$	$v_1 \rightarrow v_1$	$M_1 \rightarrow C_1$	$B_1 \rightarrow 1/R_1$	$K_1 \rightarrow 1/L_1$
$f_2(t) \rightarrow i_2(t)$	$v_2 \rightarrow v_2$	$M_2 \rightarrow C_2$	$B_3 \rightarrow 1/R_3$	$K_2 \rightarrow 1/L_2$
	$v_3 \rightarrow v_3$	$M_3 \rightarrow C_3$		$K_3 \rightarrow 1/L_3$

The node basis equations using Kirchoff's current law for the circuit shown in fig 9. are given below. (Refer fig 10, 11,12).

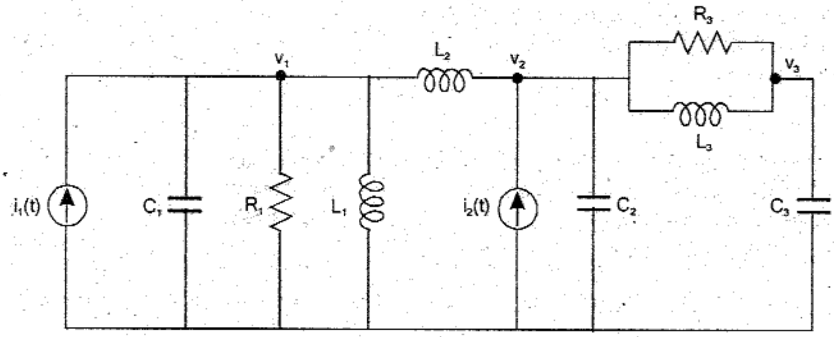


Fig 9 : Force-current electrical analogous circuit.

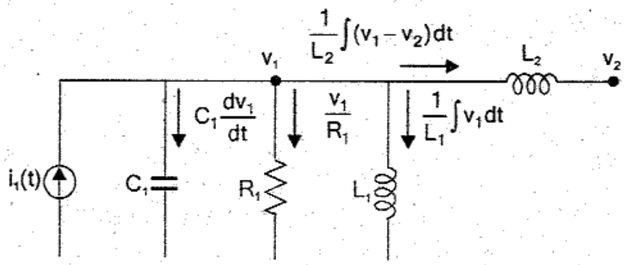


Fig 10.

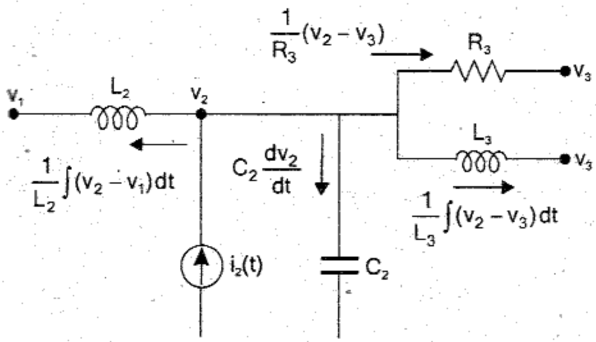


Fig 11.

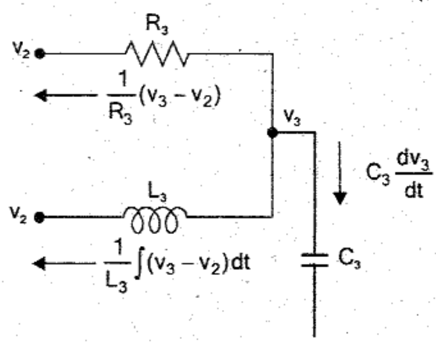


Fig 12.

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1 dt+\frac{1}{L_2}\int(v_1-v_2)dt=i_1(t) \qquad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_3}(v_2-v_3)+\frac{1}{L_3}\int(v_2-v_3)dt+\frac{1}{L_2}\int(v_2-v_1)dt=i_2(t) \qquad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_2)+\frac{1}{L_3}\int(v_3-v_2)dt=0 \qquad .....(12)$$

It is observed that node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1-10

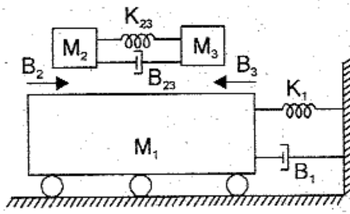


Fig 1.

Write the differential equations governing the mechanical system shown in fig 1.Draw force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

SOLUTION

The given mechanical system has three nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacements of masses M_1, M_2 and M_3 be x_1, x_2 and x_3 respectively. The corresponding velocities be v_1, v_2 and v_3.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{b1}, f_{k1}, f_{b2}, f_{b3}, and f_{m1}.

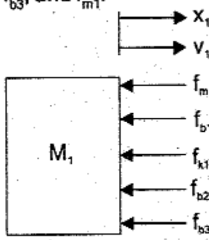


Fig 2.

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2} \quad ; \quad f_{b1} = B_1 \frac{dx_1}{dt} \quad ; \quad f_{k1} = K_1 x_1$$

$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x_2) \; ; \quad f_{b3} = B_3 \frac{d}{dt}(x_1 - x_3)$$

By Newton's second law, $f_{m1} + f_{b1} + f_{k1} + f_{b2} + f_{b3} = 0$

$$M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 x_1 + B_2 \frac{d}{dt}(x_1 - x_2) + B_3 \frac{d}{dt}(x_1 - x_3) = 0 \qquad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b23} and f_{k23}.

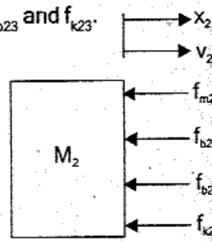


Fig 3.

$$f_{m2} = M_2 \frac{d^2x_2}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{d}{dt}(x_2 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_2 - x_3) \; ; \quad f_{k23} = K_{23}(x_2 - x_3)$$

By Newton's second law, $f_{m2} + f_{b2} + f_{b23} + f_{k23} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{d}{dt}(x_2 - x_1) + B_{23} \frac{d}{dt}(x_2 - x_3) + K_{23}(x_2 - x_3) = 0 \qquad ......(2)$$

The free body diagram of M_3 is shown in fig 4. The opposing forces are marked as f_{m3}, f_{b3}, f_{b23}, and f_{k23}.

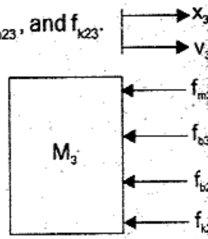


Fig 4.

$$f_{m3} = M_3 \frac{d^2x_3}{dt^2} \quad ; \quad f_{b3} = B_3 \frac{d}{dt}(x_3 - x_1)$$

$$f_{b23} = B_{23} \frac{d}{dt}(x_3 - x_2) \; ; \quad f_{k23} = K_{23}(x_3 - x_2)$$

By Newton's second law, $f_{m3} + f_{b3} + f_{b23} + f_{k23} = 0$

$$M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B_{23} \frac{d}{dt}(x_3 - x_2) + K_{23}(x_3 - x_2) = 0 \qquad .....(3)$$

On replacing the displacements by velocity in the differential equations (1), (2) and (3) governing the mechanical syster we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2} = \frac{dv}{dt}, \quad \frac{dx}{dt} = v \text{ and } x = \int v\,dt \right)$$

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_1 \int v_1 dt + B_2(v_1 - v_2) + B_3(v_1 - v_3) = 0 \qquad .....(4$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B_{23}(v_2 - v_3) + K_{23} \int (v_2 - v_3) dt = 0 \qquad .....(5$$

$$M_3 \frac{dv_3}{dt} + B_3(v_3 - v_1) + B_{23}(v_3 - v_2) + K_{23} \int (v_3 - v_2) dt = 0 \qquad .....(6$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-voltage analogous electrical circuit will have three meshes.

The elements M_1, K_1, B_1, B_3 and B_2 are connected to first node. Hence they are represented by analogous elements in mesh-1 forming a closed path. The elements M_2, K_{23}, B_{23} and B_2 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path. The elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements in mesh-3 forming a closed path.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element as common elements between mesh-2 and 3. The element B_2 is common between node-1 and 2 and so it is represented by analogous element as common element between mesh-1 and 2. The element B_3 is common between node-1 and 3 and so it is represented by analogous element between mesh-1 and 3. The force-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \to i_1$	$M_1 \to L_1$	$K_1 \to 1/C_1$ $B_2 \to R_2$
$v_2 \to i_2$	$M_2 \to L_2$	$K_{23} \to 1/C_{23}$ $B_3 \to R_3$
$v_3 \to i_3$	$M_3 \to L_3$	$B_1 \to R_1$ $B_{23} \to R_{23}$

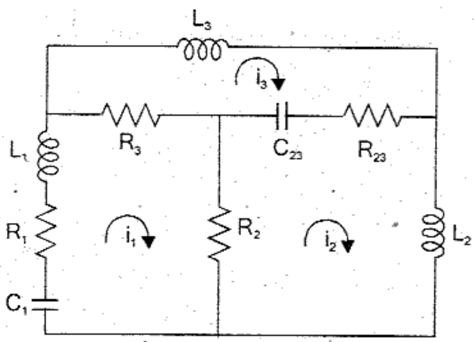


Fig 5 : Force-voltage electrical analogous circuit.

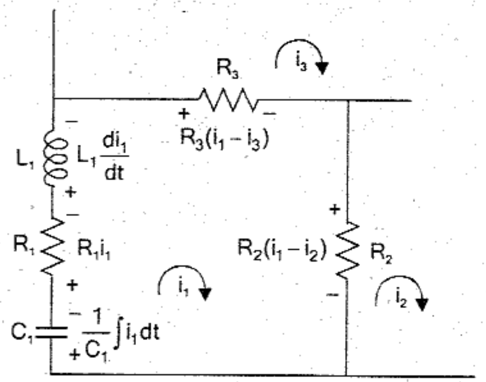


Fig 6.

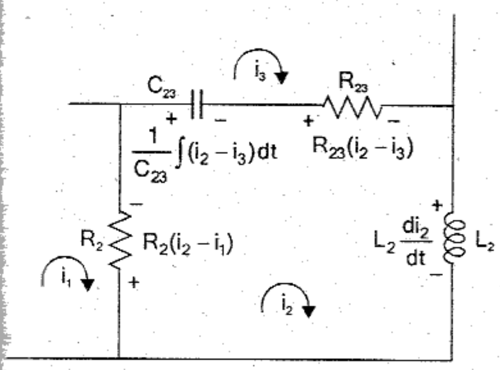


Fig 7.

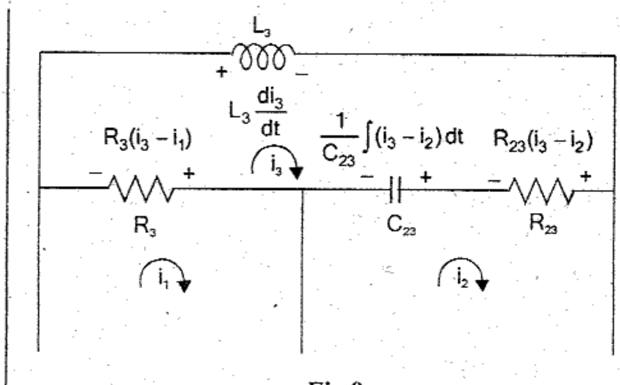


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below. (Refer fig 6, 7 & 8).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int i_1dt+R_2(i_1-i_2)+R_3(i_1-i_3)=0 \qquad .....(7)$$

$$L_2\frac{di_2}{dt}+R_2(i_2-i_1)+\frac{1}{C_{23}}\int(i_2-i_3)dt+R_{23}(i_2-i_3)=0 \qquad .....(8)$$

$$L_3\frac{di_3}{dt}+R_3(i_3-i_1)+\frac{1}{C_{23}}\int(i_3-i_2)dt+R_{23}(i_3-i_2)=0$$ .....

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and governing the mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (masses). Hence the force-current analogous electrical circuit will ha three nodes.

The elements M_1, K_1, B_1, B_2 and B_3 are connected to first node. Hence they are represented by analogous elements elements connected to node-1 in analogous electrical circuit. The elements M_2, K_{23}, B_{23} and B_2 are connected to second no Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. T elements M_3, K_{23}, B_{23} and B_3 are connected to third node. Hence they are represented by analogous elements as eleme connected to node-3 in analogous electrical circuit.

The elements K_{23} and B_{23} are common between node-2 and 3 and so they are represented by analogous element common elements between node-2 and 3 in electrical analogous circuit. The element B_2 is common between node-1 and 2 so it is represented by analogous element as common element between node-1 and 2 in electrical analogous circuit. T element B_3 is common between node-1 and 3 and so it is represented by analogous element as common element between no 1 and 3 in electrical analogous circuit. The force-current electrical analogous circuit is shown in fig 9.

The electrical analogous elements for the elements of mechanical system are given below.

$v_1 \to v_1$	$M_1 \to C_1$	$K_1 \to 1/L_1$	$B_2 \to 1/R_2$
$v_2 \to v_2$	$M_2 \to C_2$	$K_{23} \to 1/L_{23}$	$B_3 \to 1/R_3$
$v_3 \to v_3$	$M_3 \to C_3$	$B_1 \to 1/R_1$	$B_{23} \to 1/R_{23}$

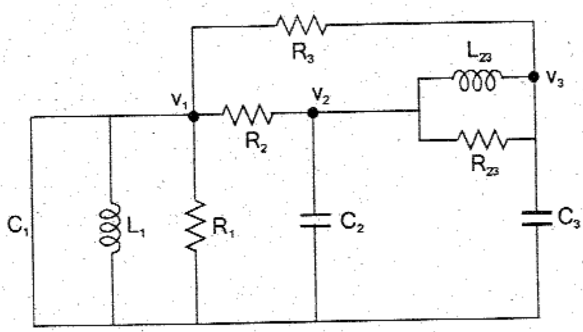


Fig 9 : Force-current electrical analogous circuit.

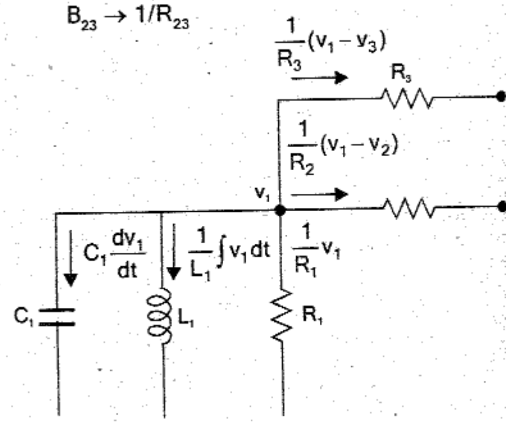


Fig 10.

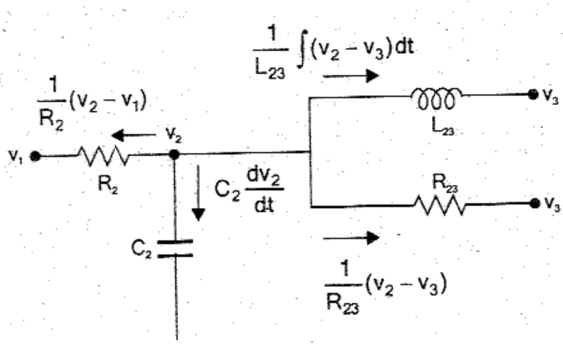


Fig 11.

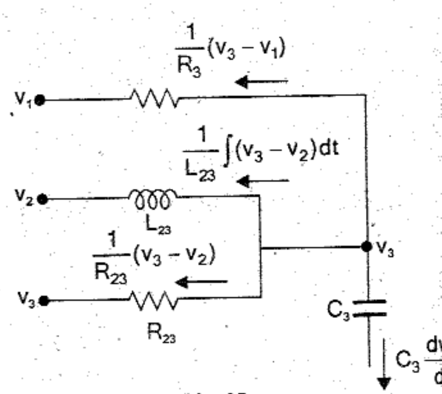


Fig 12.

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below. (Refer fig 10, 11 and 12).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int v_1 dt+\frac{1}{R_2}(v_1-v_2)+\frac{1}{R_3}(v_1-v_3)=0 \qquad .....(10)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}(v_2-v_1)+\frac{1}{L_{23}}\int(v_2-v_3)dt+\frac{1}{R_{23}}(v_2-v_3)=0 \qquad .....(11)$$

$$C_3\frac{dv_3}{dt}+\frac{1}{R_3}(v_3-v_1)+\frac{1}{L_{23}}\int(v_3-v_2)dt+\frac{1}{R_{23}}(v_3-v_2)=0 \qquad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

EXAMPLE 1.11

Write the differential equations governing the mechanical system shown in fig 1. Draw the force-voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

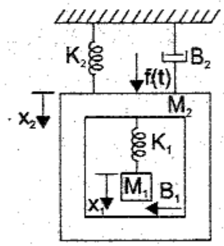


Fig 1.

SOLUTION

The given mechanical system has two nodes (masses). The differential equations governing the mechanical system are given by force balance equations at these nodes. Let the displacement of masses M_1 and M_2 be x_1 and x_2 respectively. The corresponding velocities be v_1 and v_2.

The free body diagram of M_1 is shown in fig 2. The opposing forces are marked as f_{m1}, f_{b1} and f_{k1}.

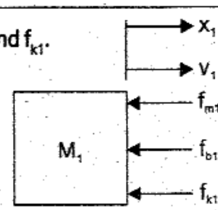


Fig 2.

$$f_{m1}=M_1\frac{d^2x_1}{dt^2}\ ;\ f_{b1}=B_1\frac{d(x_1-x_2)}{dt}\ ;\ f_{k1}=K_1(x_1-x_2)$$

By Newton's second law, $f_{m1}+f_{b1}+f_{k1}=0$

$$M_1\frac{d^2x_1}{dt^2}+B_1\frac{d(x_1-x_2)}{dt}+K_1(x_1-x_2)=0 \qquad .....(1)$$

The free body diagram of M_2 is shown in fig 3. The opposing forces are marked as f_{m2}, f_{b2}, f_{b1}, f_{k2} and f_{k1}.

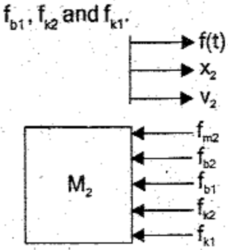


Fig 3.

$$f_{m2}=M_2\frac{d^2x_2}{dt^2}\ ;\ f_{b2}=B_2\frac{dx_2}{dt}\ ;\ f_{b1}=B_1\frac{d}{dt}(x_2-x_1)$$

$$f_{k2}=K_2x_2\ ;\ f_{k1}=K_1(x_2-x_1)$$

By Newton's second law, $f_{m2}+f_{b2}+f_{k2}+f_{b1}+f_{k1}=f(t)$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_1\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=f(t) \qquad .....(2)$$

On replacing the displacements by velocity in the differential equations (1) and (2) governing the mechanical system we get,

$$\left(\text{i.e., } \frac{d^2x}{dt^2}=\frac{dv}{dt},\ \ \frac{dx}{dt}=v \text{ and } x=\int v\,dt\right)$$

$$M_1\frac{dv_1}{dt}+B_1(v_1-v_2)+K_1\int(v_1-v_2)dt=0 \qquad .....(3)$$

$$M_2\frac{dv_2}{dt}+B_2v_2+K_2\int v_2dt+B_1(v_2-v_1)+K_1\int(v_2-v_1)dt=f(t) \qquad .....(4)$$

FORCE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force voltage analogous electrical circuit will ha two meshes. The force applied to mass, M_2 is represented by a voltage source in second mesh.

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh forming a closed path. The elements M_2, K_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented analogous element in mesh 2 forming a closed path.

The elements B_1 and K_1 are common between node 1 and 2 and so they are represented as common elements betwe mesh 1 and 2. The force-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical system are given below.

$$
\begin{array}{lllll}
f(t) \rightarrow e(t) & v_1 \rightarrow i_1 & M_1 \rightarrow L_1 & K_1 \rightarrow 1/C_1 & B_1 \rightarrow R_1 \\
 & v_2 \rightarrow i_2 & M_2 \rightarrow L_2 & K_2 \rightarrow 1/C_2 & B_2 \rightarrow R_2
\end{array}
$$

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4. are given below, (refer fig 5 and 6

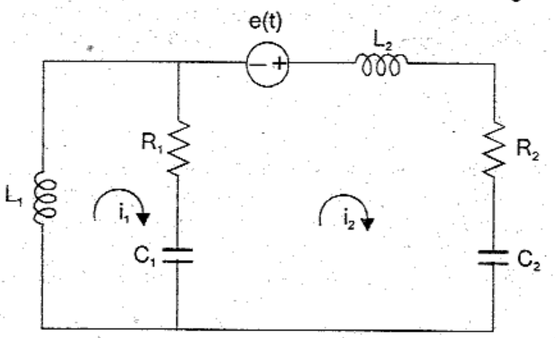


Fig 4 : Force-voltage electrical analogous circuit.

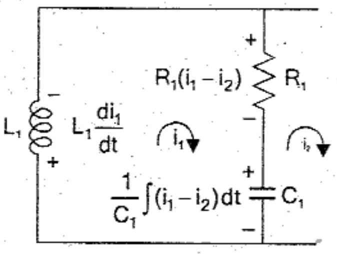


Fig 5.

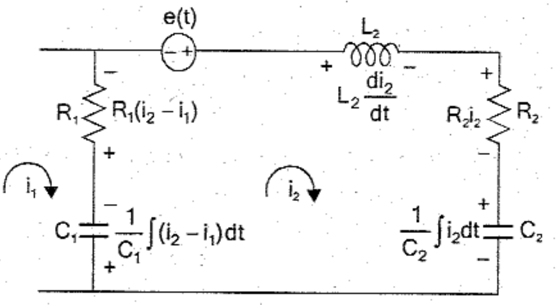


Fig 6.

$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)\,dt=0 \qquad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2 i_2+\frac{1}{C_2}\int i_2\,dt+\frac{1}{C_1}\int(i_2-i_1)\,dt+R_1(i_2-i_1)=e(t) \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing th mechanical system.

FORCE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will hav two nodes. The force applied to mass M_2 is represented as a current source connected to node-2 in analogous electrical circu

The elements M_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous elements a elements connected to node-1 in analogous electrical circuit. The elements M_2, K_2, B_2, B_1 and K_1 are connected to secon node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit.

The elements K_1 and B_1 is common to node-1 and 2 and so they are represented by analogous element as common elements between two nodes in analogous circuit. The force-current electrical analogous circuit is shown in fig 7.

The electrical analogous elements for the elements of mechanical system are given below.

$f(t) \rightarrow i(t)$ $\quad v_1 \rightarrow v_1$ $\quad M_1 \rightarrow C_1$ $\quad B_1 \rightarrow 1/R_1$ $\quad K_1 \rightarrow 1/L_1$

$v_2 \rightarrow v_2$ $\quad M_2 \rightarrow C_2$ $\quad B_2 \rightarrow 1/R_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig.7, are given below, (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)\,dt=0 \qquad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_1}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=i(t) \qquad .....(8)$$

It is observed that the node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

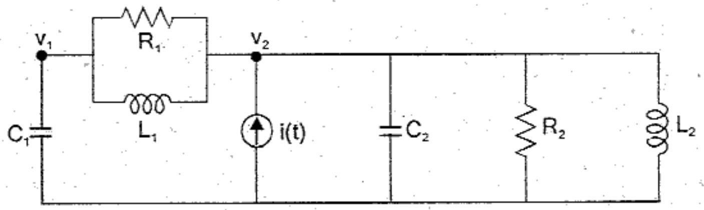


Fig 7 : Force-current electrical analogous circuit.

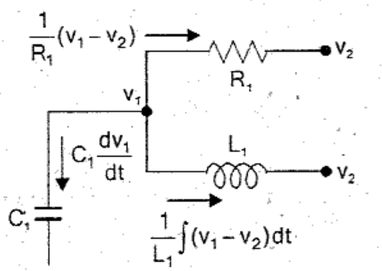


Fig 8.

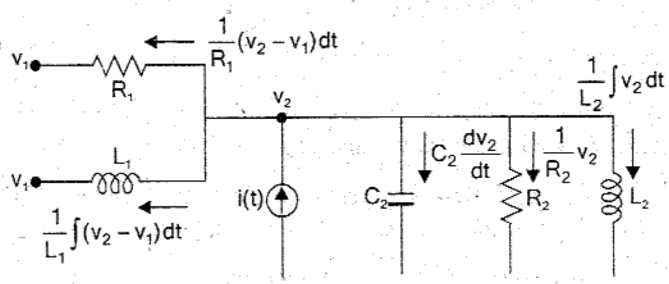


Fig 9.

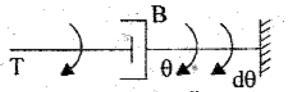
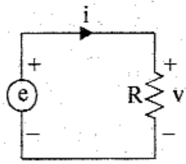
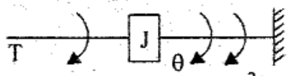
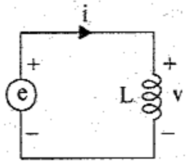
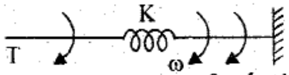
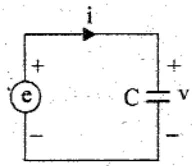
1.10 ELECTRICAL ANALOGOUS OF MECHANICAL ROTATIONAL SYSTEMS

The three basic elements moment of inertia, rotational dashpot and torsional spring that are used in modelling mechanical rotational systems are analogous to resistance, inductance and capacitance of electrical systems. The input torque in mechanical system is analogous to either voltage source or current source in electrical systems. The output angular velocity (first derivative of angular displacement) in mechanical rotational system is analogous to either current or voltage in an element in electrical system. Since the electrical systems has two types of inputs either voltage source or current source, there are two types of analogies: ***torque-voltage analogy and torque-current analogy***.

TORQUE-VOLTAGE ANALOGY

The torque balance equations of mechanical rotational elements and their analogous electrical elements in torque-voltage analogy are shown in table-1.6. The table-1.7 shows the list of analogous quantities in torque-voltage analogy.

TABLE-1.6 : Analogous Element of Torque-Voltage Analogy

Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Voltage source Output : Current through the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $e = v\ ;\ v = Ri$ $\therefore e = Ri$
 $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$ $\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$	 $e = v\ ;\ v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $\theta = \int \omega\, dt$ $T = K\theta = K\int \omega\, dt$	 $e = v\ ;\ v = \frac{1}{C}\int i\, dt$ $\therefore e = \frac{1}{C}\int i\, dt$

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on torque-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same angular velocity are said to be in series.
2. The elements having same angular velocity in mechanical system should have analogous same current in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. The moment of inertia of mass is considered as a node.
4. The number of meshes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node (moment of inertia of mass) in mechanical system should be represented by analogous element in a closed loop in analogous electrical system.
6. The element connected between two nodes (moment of inertia) in mechanical system is represented as a common element between two meshes in electrical analogous system.

TABLE-1.7 : Analogous Quantities in Torque-Voltage Analogy

Item	Mechanical rotational system	Electrical system (mesh basis system)
Independent variable (input)	Torque, T	Voltage, e, v
Dependent variable (output)	Angular Velocity, ω	Current, i
	Angular displacement, θ	Charge, q
Dissipative element	Rotational coefficient of dashpot, B	Resistance, R
Storage element	Moment of inertia, J	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff's voltage law $\Sigma v = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

TORQUE-CURRENT ANALOGY

The torque balance equations of mechanical elements and their analogous electrical elements in torque-current analogy are shown in table-1.8. The table-1.9 shows the list of analogous quantities in torque-current analogy.

The following points serve as guidelines to obtain electrical analogous of mechanical rotational systems based on Torque-current analogy.

1. In electrical systems the elements in parallel will have same voltage, likewise in mechanical systems, the elements having same torque are said to be in parallel.
2. The elements having same angular velocity in mechanical system should have analogous same voltage in electrical analogous system.
3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. The moment of inertia of mass is considered as a node.
4. The number of nodes in electrical analogous is same as that of the number of nodes (moment of inertia of mass) in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of angular velocities of nodes (moment of inertia of mass) in mechanical system.
5. The mechanical driving sources (Torque) and passive elements connected to the node in mechanical system should be represented by analogous element connected to a node in analogous electrical system.
6. The element connected between two nodes (moment of inertia of mass) in mechanical system is represented as a common element between two nodes in electrical analogous system.

TABLE-1.8 : Analogous Elements in Torque-Current Analogy

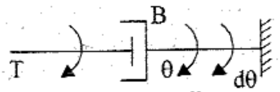
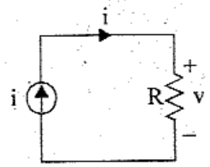
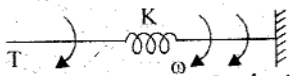
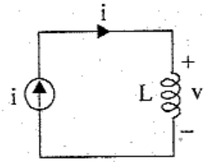
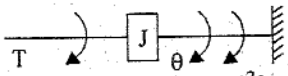
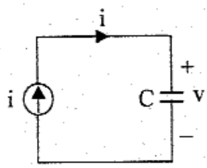
Mechanical rotational system	Electrical system
Input : Torque Output : Angular velocity	Input : Current source Output : Voltage across the element
 $\omega = \frac{d\theta}{dt}$ $T = B\frac{d\theta}{dt} = B\omega$	 $i = \frac{1}{R}v$
 $\theta = \int \omega\, dt$ $T = K\theta = K\int \omega\, dt$	 $i = \frac{1}{L}\int v\, dt$
 $\propto = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$ $T = J\frac{d^2\theta}{dt^2} = J\frac{d\omega}{dt}$	 $i = C\frac{dv}{dt}$

TABLE-1.9 : Analogous Quantities in Torque-Current Analogy

Item	Mechanical rotational system	Electrical system (node basis system)
Independent variable (input)	Torque, T	Current, i
Dependent variable (output)	Angular Velocity, ω	Voltage, v
	Angular displacement, θ	Flux, ϕ
Dissipative element	Rotational frictional coefficient of dashpot, B	Conductance, G = 1/R
Storage element	Moment of inertia, J	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\Sigma T = 0$	Kirchoff 's current law $\Sigma i = 0$
Changing the level of independent variable	Gear $\frac{T_1}{T_2} = \frac{n_1}{n_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

EXAMPLE 1.12

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

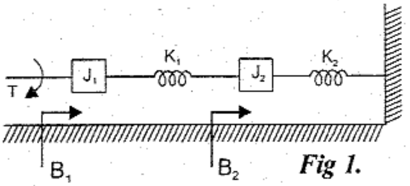


Fig 1.

SOLUTION

The given mechanical rotational system has two nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1 and J_2 be θ_1 and θ_2 respectively. The corresponding angular velocities be ω_1 and ω_2.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

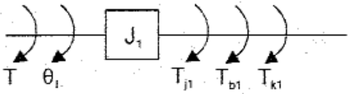


Fig 2.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b1} = B_1 \frac{d\theta_1}{dt} \quad ; \quad T_{k1} = K_1(\theta_1 - \theta_2)$$

By Newton's second law, $T_{j1} + T_{b1} + T_{k1} = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) = T \qquad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{k2} and T_{k1}.

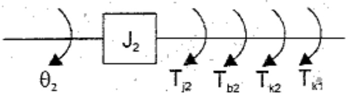


Fig 3.

$$T_{j2} = J_2 \frac{d^2\theta_2}{dt^2} \quad ; \quad T_{b2} = B_2 \frac{d\theta_2}{dt}$$

$$T_{k2} = K_2\theta_2 \quad ; \quad T_{k1} = K_1(\theta_2 - \theta_1)$$

By Newton's second law, $T_{j2} + T_{b2} + T_{k2} + T_{k1} = 0$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2) + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} \; ; \quad \frac{d\theta}{dt} = \omega \quad \text{and} \quad \theta = \int \omega \, dt\right)$$

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T \qquad .....(3)$$

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_2\int\omega_2 dt + K_1\int(\omega_2 - \omega_1)dt = 0 \qquad .....(4)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-voltage analogous electrical circuit will have two meshes. The torque applied to J_1 is represented by a voltage source in first mesh. The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements in mesh-2 forming a closed path.

The element K_1 is common between node-1 and 2 and so it is represented by analogous element as common element between two meshes. The torque-voltage electrical analogous circuit is shown in fig 4.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$ $\quad J_1 \rightarrow L_1$ $\quad B_1 \rightarrow R_1$ $\quad K_1 \rightarrow 1/C_1$

$\omega_1 \rightarrow i_1$ $\quad J_2 \rightarrow L_2$ $\quad B_2 \rightarrow R_2$ $\quad K_2 \rightarrow 1/C_2$

$\omega_2 \rightarrow i_2$

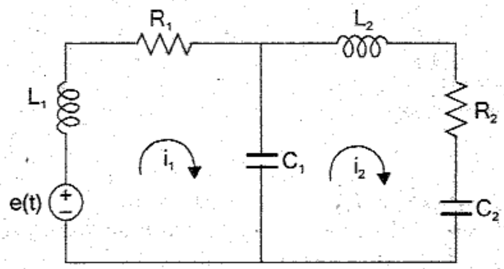


Fig 4 : Torque-voltage electrical analogous circuit.

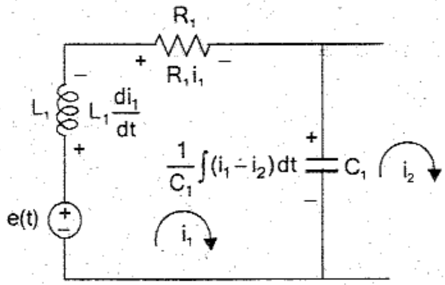


Fig 5.

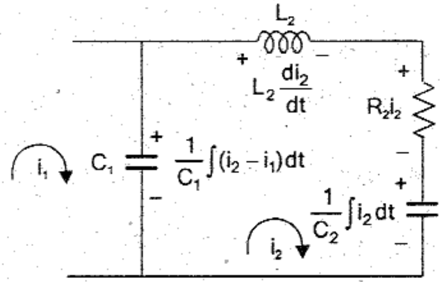


Fig 6.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 4 are given below (Refer fig 5 and 6).

$$L_1\frac{di_1}{dt}+R_1i_1+\frac{1}{C_1}\int(i_1-i_2)=e(t) \qquad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2dt+\frac{1}{C_1}\int(i_2-i_1)dt=0 \qquad .....(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has two nodes (J_1 and J_2). Hence the torque-current analogous electrical circuit will have two nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements J_1, B_1 and K_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, K_2 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit.

The element K_1 is common between node-1 and 2. So it is represented by analogous element as common element between node-1 and 2. The torque-current electrical analogous circuit is shown in fig 7.

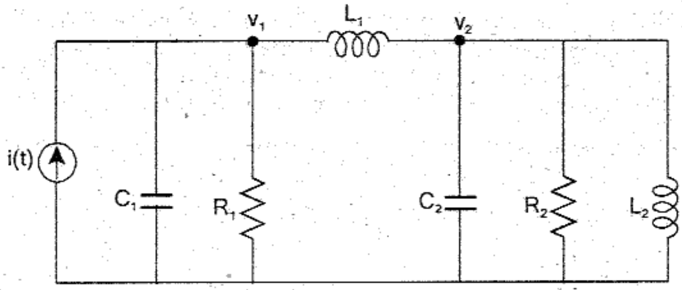


Fig 7 : Torque-current electrical analogous circuit.

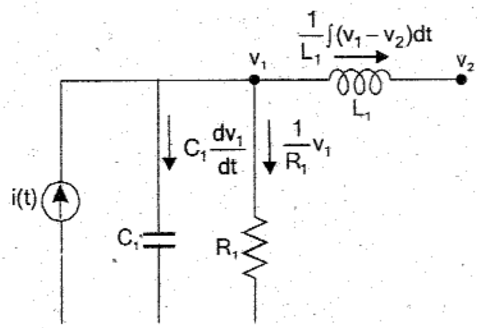


Fig 8.

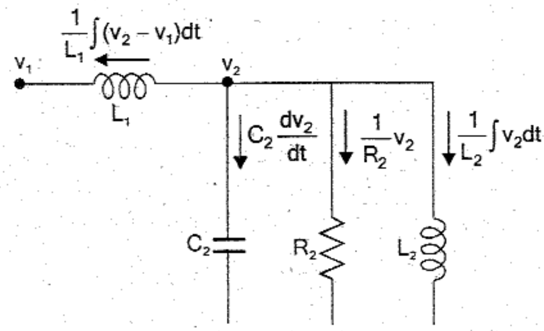


Fig 9.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\quad B_1 \rightarrow 1/R_1$ $\quad \omega_1 \rightarrow v_1$ $\quad J_1 \rightarrow C_1$ $\quad K_1 \rightarrow 1/L_1$

$B_2 \rightarrow 1/R_2$ $\quad \omega_2 \rightarrow v_2$ $\quad J_2 \rightarrow C_2$ $\quad K_2 \rightarrow 1/L_2$

The node basis equations using Kirchoff's current law for the circuit shown in fig 7 are given below (Refer fig 8 and 9).

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{L_1}\int(v_1-v_2)dt = i(t) \quad .....(7)$$

$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{L_1}\int(v_2-v_1)dt = 0 \quad .....(8)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

EXAMPLE 1.13

Write the differential equations governing the mechanical rotational system shown in fig 1. Draw the torque-voltage and torque-current electrical analogous circuits and verify by writing mesh and node equations.

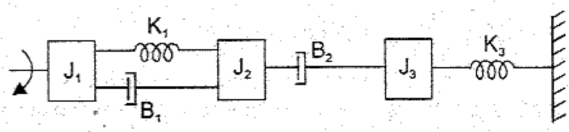


Fig 1.

SOLUTION

The given mechanical rotational system has three nodes (moment of inertia of masses). The differential equations governing the mechanical rotational system are given by torque balance equations at these nodes.

Let the angular displacements of J_1, J_2 and J_3 be θ_1, θ_2 and θ_3 respectively. The corresponding angular velocities be ω_1, ω_2 and ω_3.

The free body diagram of J_1 is shown in fig 2. The opposing torques are marked as T_{j1}, T_{b1} and T_{k1}.

$$T_{j1} = J_1\frac{d^2\theta_1}{dt^2} \; ; \; T_{b1} = B_1\frac{d(\theta_1-\theta_2)}{dt}$$

$$T_{k1} = K_1(\theta_1-\theta_2)$$

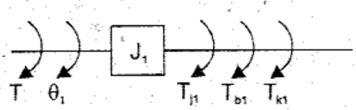


Fig 2.

By Newton's second law, $T_{j1}+T_{b1}+T_{k1}=T$

$$J_1\frac{d^2\theta_1}{dt^2}+B_1\frac{d(\theta_1-\theta_2)}{dt}+K_1(\theta_1-\theta_2) = T \quad .....(1)$$

The free body diagram of J_2 is shown in fig 3. The opposing torques are marked as T_{j2}, T_{b2}, T_{b1} and T_{k1}.

$$T_{j2} = J_2\frac{d^2\theta_2}{dt^2} \; ; \qquad T_{b2} = B_2\frac{d(\theta_2-\theta_3)}{dt}$$

$$T_{k1} = K_1(\theta_2 - \theta_1)\ ;\ \ T_{b1} = B_1\frac{d(\theta_2 - \theta_1)}{dt}$$

By Newton's second law, $T_{j2} + T_{b2} + T_{b1} + T_{k1} = 0$

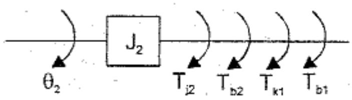


Fig 3.

$$J_2\frac{d^2\theta_2}{dt^2} + B_2\frac{d(\theta_2 - \theta_3)}{dt} + B_1\frac{d(\theta_2 - \theta_1)}{dt} + K_1(\theta_2 - \theta_1) = 0 \qquad .....(2)$$

The free body diagram of J_3 is shown in fig 4. The opposing torques are marked as T_{j3}, T_{b2}, and T_{k3}.

$$T_{j3} = J_3\frac{d^2\theta_3}{dt^2}\ ;\ \ T_{b2} = B_2\frac{d(\theta_3 - \theta_2)}{dt}\ ;\ \ T_{k3} = K_3\theta_3$$

By Newton's second law, $T_{j3} + T_{b2} + T_{k3} = 0$

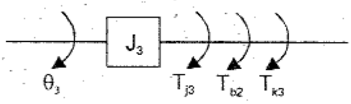


Fig 4.

$$\therefore J_3\frac{d^2\theta_3}{dt^2} + B_2\frac{d(\theta_3 - \theta_2)}{dt} + K_3\theta_3 = 0 \qquad .....(3)$$

On replacing the angular displacements by angular velocity in the differential equations (1) and (2) governing the mechanical rotational system we get,

$$\left(\text{i.e., } \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}\ ;\ \ \frac{d\theta}{dt} = \omega \ \text{ and } \ \theta = \int \omega\, dt\right)$$

$$J_1\frac{d\omega_1}{dt} + B_1(\omega_1 - \omega_2) + K_1\int(\omega_1 - \omega_2)\,dt = T \qquad .....(4)$$

$$J_2\frac{d\omega_2}{dt} + B_1(\omega_2 - \omega_1) + B_2(\omega_2 - \omega_3) + K_1\int(\omega_2 - \omega_1)\,dt = 0 \qquad .....(5)$$

$$J_3\frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3\int\omega_3 dt = 0 \qquad .....(6)$$

TORQUE-VOLTAGE ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-voltage analogous electrical circuit will have three meshes. The torque applied to J_1 is represented by a voltage source in first mesh.

The elements J_1, K_1 and B_1 are connected to first node. Hence they are represented by analogous element in mesh-1 forming a closed path. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous element in mesh-2 forming a closed path. The element J_3, B_2 and K_3 are connected to third node. Hence they are represented by analogous element in mesh-3 forming a closed path.

The elements K_1 and B_1 are common between the nodes-1 and 2 and so they are represented by analogous element as common between mesh-1 and 2. The element B_2 is common between the nodes-2 and 3 and so it is represented by analogous element as common element between the mesh-2 and 3. The torque-voltage electrical analogous circuit is shown in fig 5.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow e(t)$	$\omega_1 \rightarrow i_1$	$J_1 \rightarrow L_1$	$B_1 \rightarrow R_1$	$K_1 \rightarrow 1/C_1$
	$\omega_2 \rightarrow i_2$	$J_2 \rightarrow L_2$	$B_2 \rightarrow R_2$	$K_3 \rightarrow 1/C_3$
	$\omega_3 \rightarrow i_3$	$J_3 \rightarrow L_3$		

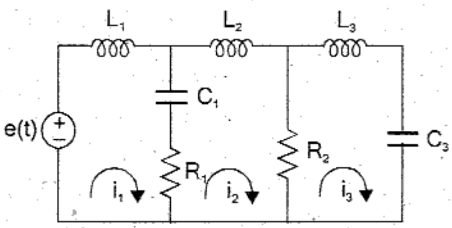


Fig 5 : Torque-voltage electrical analogous circuit.

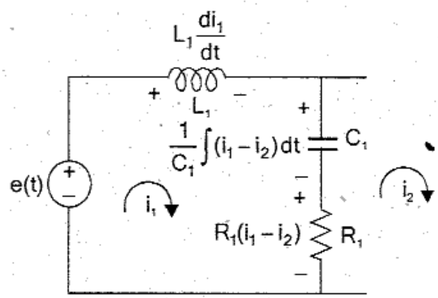


Fig 6.

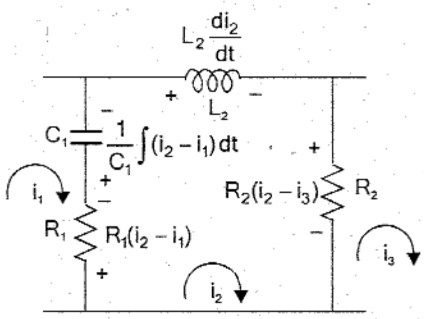


Fig 7.

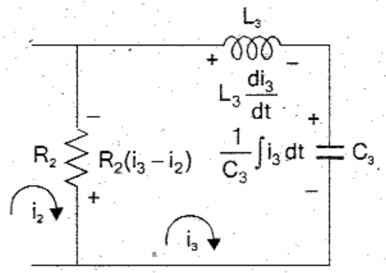


Fig 8.

The mesh basis equations using Kirchoff's voltage law for the circuit shown in fig 5 are given below (Refer fig 6, 7 and 8).

$$L_1\frac{di_1}{dt}+R_1(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \quad .....(7)$$

$$L_2\frac{di_2}{dt}+R_1(i_2-i_1)+R_2(i_2-i_3)+\frac{1}{C_1}\int(i_2-i_1)dt=0 \quad .....(8)$$

$$L_3\frac{di_3}{dt}+R_2(i_3-i_2)+\frac{1}{C_3}\int i_3dt=0 \quad .....(9)$$

It is observed that the mesh basis equations (7), (8) and (9) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

TORQUE-CURRENT ANALOGOUS CIRCUIT

The given mechanical system has three nodes (J_1, J_2 and J_3). Hence the torque-current analogous electrical circuit will have three nodes. The torque applied to J_1 is represented as a current source connected to node-1 in analogous electrical circuit.

The elements K_1, J_1 and B_1 are connected to first node. Hence they are represented by analogous elements as elements connected to node-1 in analogous electrical circuit. The elements J_2, B_2, B_1 and K_1 are connected to second node. Hence they are represented by analogous elements as elements connected to node-2 in analogous electrical circuit. The elements J_3, B_2, and K_3 are connected to third node. Hence they are represented by analogous elements as elements connected to node-3 in analogous electrical circuit.

The elements K_1 and B_1 are common between node-1 and 2 and so they are represented by analogous element as common elements between node-1 and 2. The element B_2 is common between node-2 and 3 and so it is represented as common element between node-2 and 3 in analogous circuit. The torque-current electrical analogous circuit is shown in fig 9.

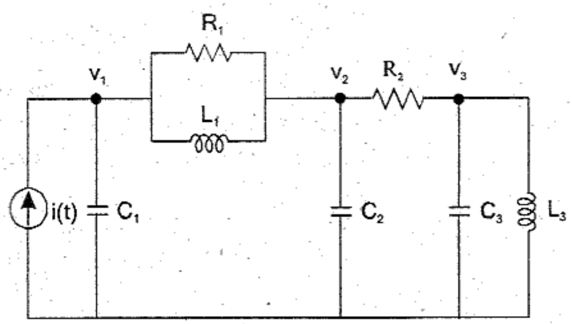


Fig 9 : Torque-current electrical analogous circuit.

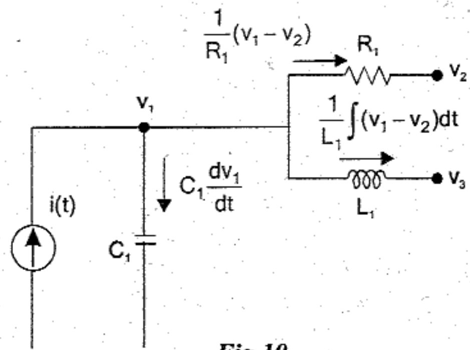


Fig 10.

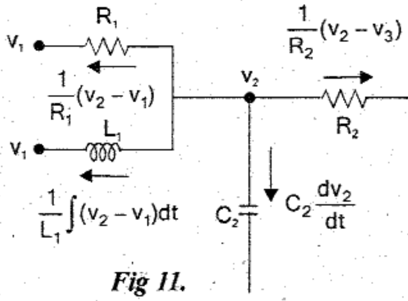


Fig 11.

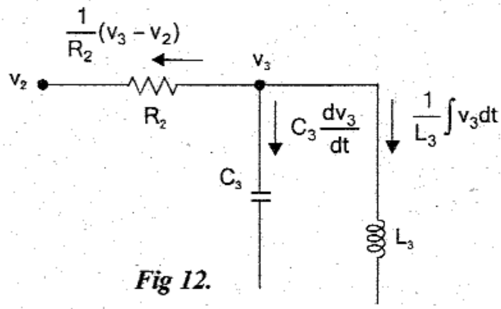


Fig 12.

The electrical analogous elements for the elements of mechanical rotational system are given below.

$T \rightarrow i(t)$ $\omega_1 \rightarrow v_1$ $J_1 \rightarrow C_1$ $B_1 \rightarrow 1/R_1$ $K_1 \rightarrow 1/L_1$

$\omega_2 \rightarrow v_2$ $J_2 \rightarrow C_2$ $B_2 \rightarrow 1/R_2$ $K_3 \rightarrow 1/L_3$

$\omega_3 \rightarrow v_3$ $J_3 \rightarrow C_3$

The node basis equations using Kirchoff's current law for the circuit shown in fig 9 are given below (Refer fig 10, 11 and 12).

$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t) \qquad .....(10)$$

$$C_2\frac{dv_2}{dt} + \frac{1}{R_1}(v_2 - v_1) + \frac{1}{R_2}(v_2 - v_3) + \frac{1}{L_1}\int(v_2 - v_1)\,dt = 0 \qquad .....(11)$$

$$C_3\frac{dv_3}{dt} + \frac{1}{R_2}(v_3 - v_2) + \frac{1}{L_3}\int v_3\,dt = 0 \qquad .....(12)$$

It is observed that the node basis equations (10), (11) and (12) are similar to the differential equations (4), (5) and (6) governing the mechanical system.

1.11 BLOCK DIAGRAMS

A control system may consist of a number of components. In control engineering to show the functions performed by each component, we commonly use a diagram called the block diagram. A ***block diagram*** of a system is a pictorial representation of the functions performed by each component and of the flow of signals. Such a diagram depicts the interrelationships that exist among the various components. The elements of a block diagram are ***block***, ***branch point*** and ***summing point.***

BLOCK

In a block diagram all system variables are linked to each other through functional blocks. The ***functional block*** or simply ***block*** is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Figure 1.25 shows the block diagram of functional block.

The arrowhead pointing towards the block indicates the input, and the arrowhead leading away from the block represents the output. Such arrows are referred to as signals. The output signal from the block is given by the product of input signal and transfer function in the block.

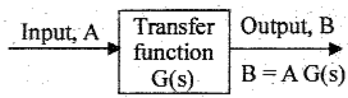


***Fig 1.25 :** Functional block.*

SUMMING POINT

Summing points are used to add two or more signals in the system. Referring to figure 1.26, a circle with a cross is the symbol that indicates a summing operation.

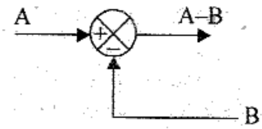


Fig 1.26 : Summing point.

The plus or minus sign at each arrowhead indicates whether the signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

BRANCH POINT

A ***branch point*** is a point from which the signal from a block goes concurrently to other blocks or summing points.

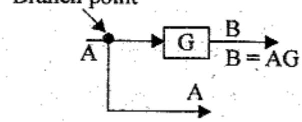


Fig 1.27 : Branch point.

CONSTRUCTING BLOCK DIAGRAM FOR CONTROL SYSTEMS

A control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram. By taking Laplace transform the differential equations are converted to algebraic equations. The equations will have variables and constants. From the working knowledge of the system the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram. The output of one section will be input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

EXAMPLE 1.14

Construct the block diagram of armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7),

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad .....(1)$$

$$T = K_t i_a \quad .....(2)$$

$$T = J\frac{d\omega}{dt} + B\omega \quad .....(3)$$

$$e_b = K_b\,\omega \quad .....(4)$$

$$\omega = \frac{d\theta}{dt} \quad .....(5)$$

On taking Laplace transform of equation (1) we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \quad .....(6)$$

In equation (6), $V_a(s)$ and $E_b(s)$ are inputs and $I_a(s)$ is the output. Hence the equation (6) is rearranged and the block diagram for this equation is shown in fig 1.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\,[V_a(s) - E_b(s)]$$

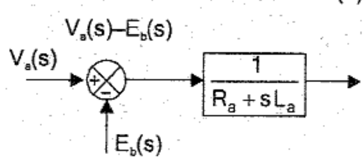


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_t\,I_a(s) \quad .....(7)$$

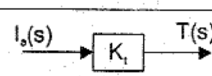


Fig 2.

In equation (7), $I_a(s)$ is the input and $T(s)$ is the output. The block diagram for this equation is shown in fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = Js\,\omega(s) + B\,\omega(s) \qquad .....(8)$$

In equation (8), T(s) is the input and $\omega(s)$ is the output. Hence the equation (8) is rearranged and the block diagram f
this equation is shown in fig (3).

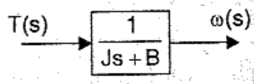


Fig 3.

$$T(s) = (Js + B)\,\omega(s)$$

$$\therefore \omega(s) = \frac{1}{Js+B}\,T(s)$$

On taking Laplace transform of equation (4) we get,

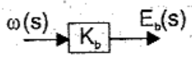


Fig 4.

$$E_b(s) = K_b\,\omega(s) \qquad .....(9)$$

In equation (9), $\omega(s)$ is the input and $E_b(s)$ is the output. The block diagram for this equation is shown in fig 4.

On taking Laplace transform of equation (5) we get,

$$\omega(s) = s\,\theta(s) \qquad .....(10)$$

In equation (10), $\omega(s)$ is the input and $\theta(s)$ is the output. Hence equation (10) is rearranged and the block diagram for th
equation is shown in fig 5.

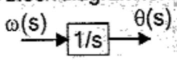


Fig 5.

$$\theta(s) = \frac{1}{s}\,\omega(s)$$

The overall block diagram of armature controlled dc motor is obtained by connecting the various sections shown in fig 1
fig 5. The overall block diagram is shown in fig 6.

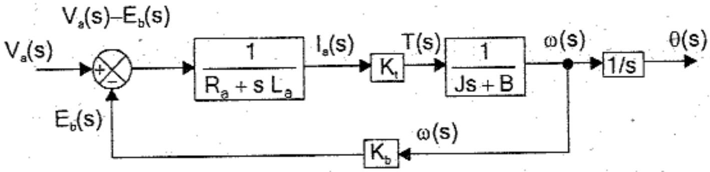


Fig 6 : Block diagram of armature controlled dc motor.

EXAMPLE 1.15

Construct the block diagram of field controlled dc motor.

SOLUTION

The differential equations governing the field controlled dc motor are (refer section 1.8),

$$v_f = R_f\, i_f + L_f\,\frac{di_f}{dt} \qquad .....$$

$$T = K_{tf}\, i_f \qquad .....$$

$$T = J\,\frac{d^2\theta}{dt^2} + B\,\frac{d\theta}{dt} \qquad .....$$

On taking Laplace transform of equation (1) we get,

$$V_f(s) = R_f\, I_f(s) + L_f\, s\, I_f(s) \qquad .....$$

In equation (4), $V_f(s)$ is the input and $I_f(s)$ is the output. Hence the equation (4) is rearranged and the block diagram for is equation is shown in fig 1.

$$V_f(s) = I_f(s)\left[R_f + sL_f\right]$$

$$\therefore I_f(s) = \frac{1}{R_f + s\,L_f}\,V_f(s)$$

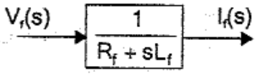


Fig 1.

On taking Laplace transform of equation (2) we get,

$$T(s) = K_{tf}\, I_f(s) \qquad .....(5)$$

In equation (5), $I_f(s)$ is the input and $T(s)$ is the output. The block diagram for this equation is shown in fig 2.

Fig 2.

On taking Laplace transform of equation (3) we get,

$$T(s) = J\,s^2\theta(s) + B\,s\,\theta(s) \qquad .....(6)$$

In equation (6), $T(s)$ is input and $\theta(s)$ is the output. Hence equation (6) is rearranged and the block diagram for this equation is shown in fig 3.

$$T(s) = (J\,s^2 + Bs)\,\theta(s)$$

$$\therefore \theta(s) = \frac{1}{Js^2 + Bs}\,T(s)$$

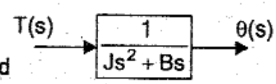


Fig 3.

The overall block diagram of field controlled dc motor is obtained by connecting the various section shown in fig 1 to 3. The overall block diagram is shown in fig 4.

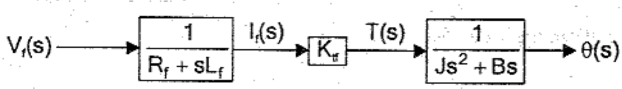


Fig 4 : Block diagram of field controlled dc motor.

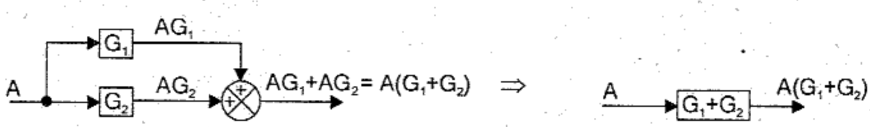
BLOCK DIAGRAM REDUCTION

The block diagram can be reduced to find the overall transfer function of the system. The following rules can be used for block diagram reduction. The rules are framed such that any modification made on the diagram does not alter the input-output relation.

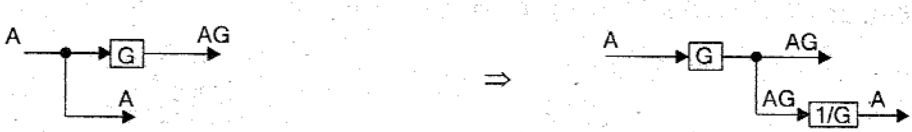
RULES OF BLOCK DIAGRAM ALGEBRA

Rule-1 : ***Combining the blocks in cascade***

Rule-2 : ***Combining Parallel blocks (or combining feed forward paths)***

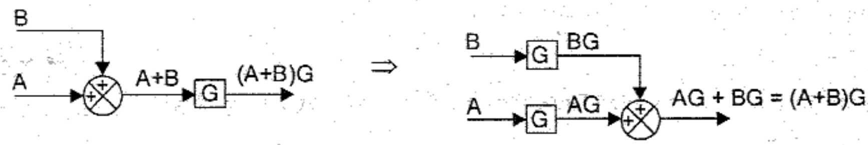


Rule-3 : ***Moving the branch point ahead of the block***

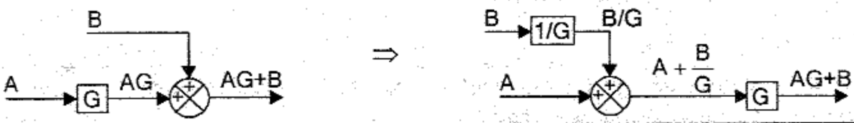


Rule-4 : *Moving the branch point before the block*

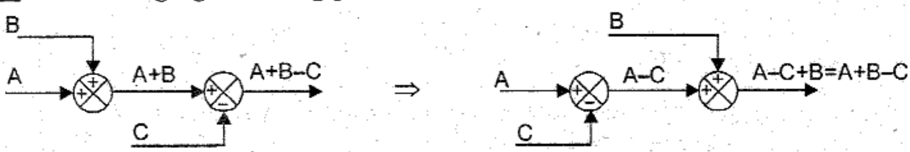
Rule-5 : *Moving the summing point ahead of the block*



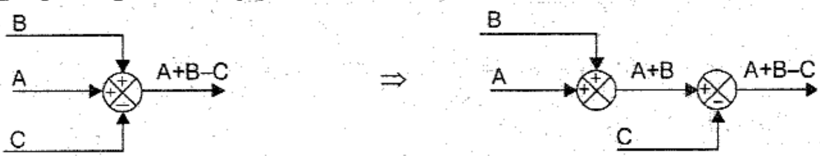
Rule-6 : *Moving the summing point before the block*



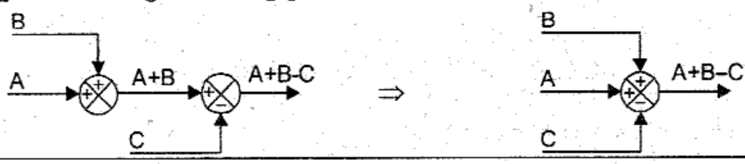
Rule-7 : *Interchanging summing point*



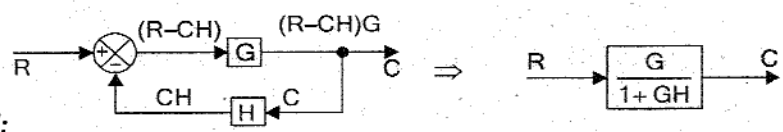
Rule-8 : *Splitting summing points*



Rule-9 : *Combining summing points*



Rule-10 : *Elimination of (negative) feedback loop*

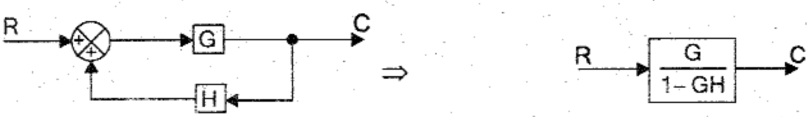


Proof:

$$C=(R-CH)G \quad \Rightarrow \quad C=RG-CHG \quad \Rightarrow \quad C+CHG=RG$$

$$\therefore C(1+HG)=RG \quad \Rightarrow \quad \frac{C}{R}=\frac{G}{1+GH}$$

Rule-11 : *Elimination of (positive) feedback loop*



EXAMPLE 1.16

Reduce the block diagram shown in fig 1 and find C/R.

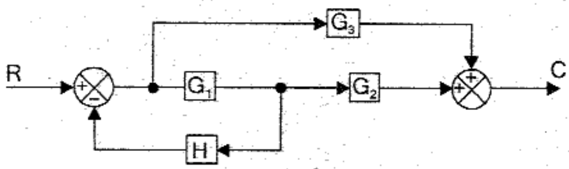
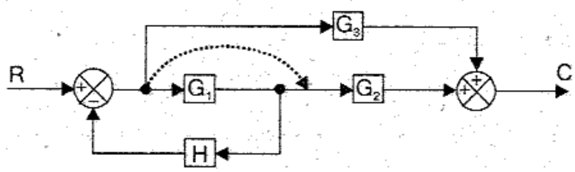


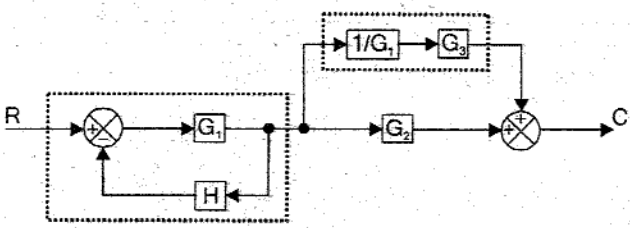
Fig 1.

SOLUTION

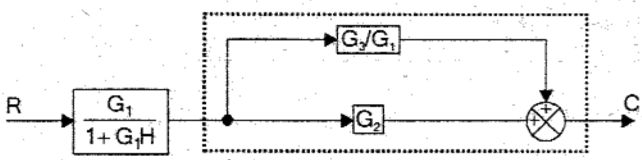
Step 1: Move the branch point after the block.



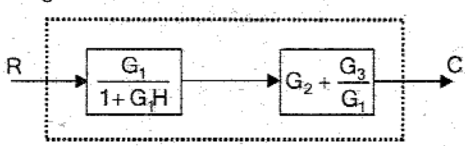
Step 2: Eliminate the feedback path and combining blocks in cascade.



Step 3: Combining parallel blocks



Step 4: Combining blocks in cascade



$$\frac{C}{R}=\left(\frac{G_1}{1+G_1H}\right)\left(G_2+\frac{G_3}{G_1}\right)=\left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_1G_2+G_3}{G_1}\right)=\frac{G_1G_2+G_3}{1+G_1H}$$

RESULT

The overall transfer function of the system, $\frac{C}{R}=\frac{G_1G_2+G_3}{1+G_1H}$

EXAMPLE 1.17

Using block diagram reduction technique find closed loop transfer function of the system whose block diagram is show in fig 1.

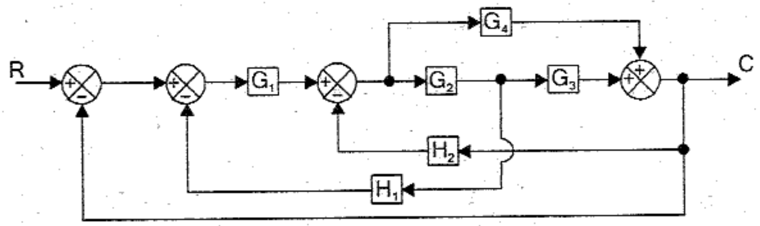
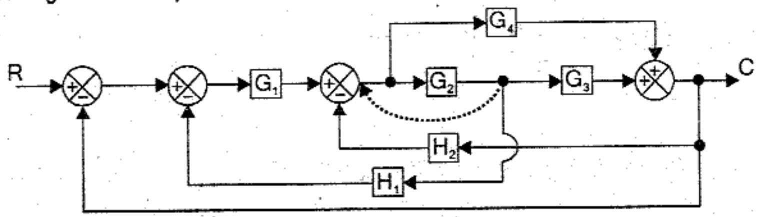


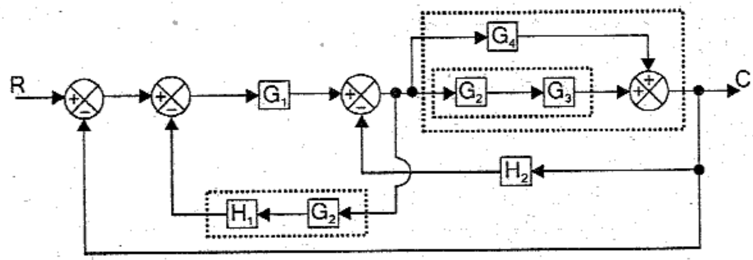
Fig 1.

SOLUTION

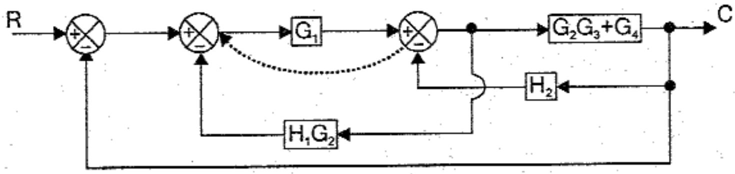
Step 1: Moving the branch point before the block



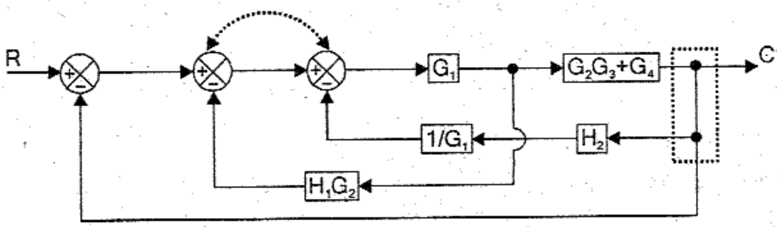
Step 2: Combining the blocks in cascade and eliminating parallel blocks



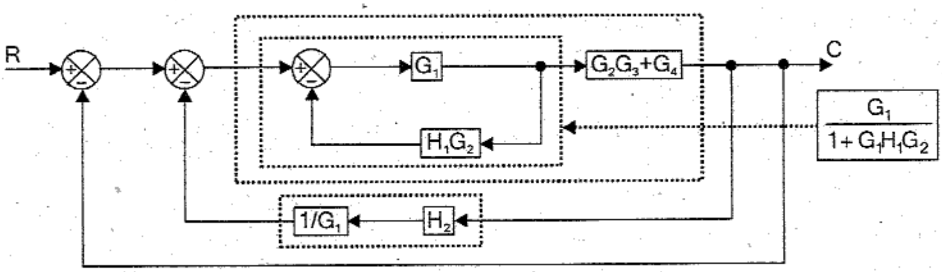
Step 3: Moving summing point before the block.



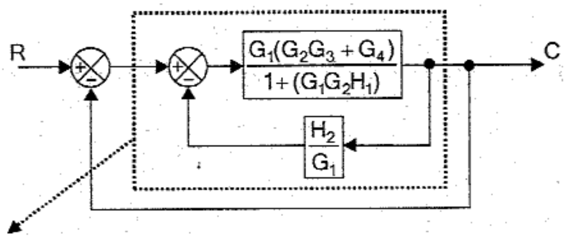
Step 4: Interchanging summing points and modifying branch points.



Step 5: Eliminating the feedback path and combining blocks in cascade

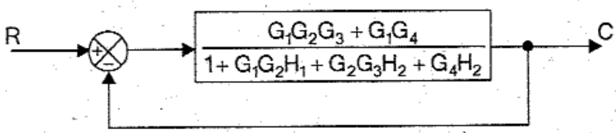


Step 6: Eliminating the feedback path



$$\frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\,\dfrac{H_2}{G_1}} \Rightarrow \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1}}{\dfrac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}} \Rightarrow \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 7: Eliminating the feedback path



$$\frac{C}{R}=\frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

RESULT

The overall transfer function is given by,

$$\frac{C}{R}=\frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

EXAMPLE 1.18

Determine the overall transfer function $\frac{C(s)}{R(s)}$ for the system shown in fig 1.

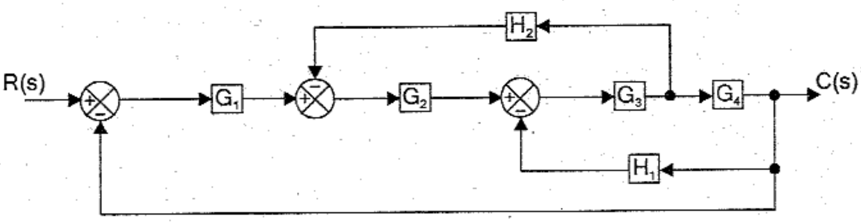
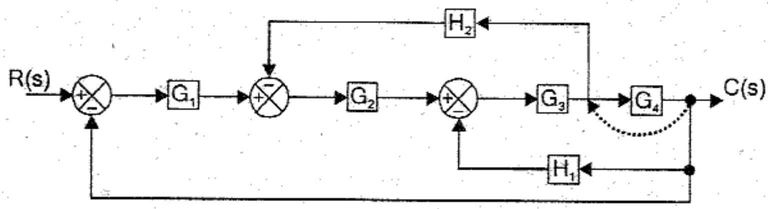


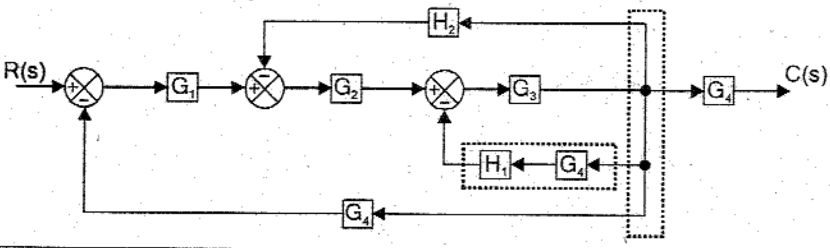
Fig 1.

SOLUTION

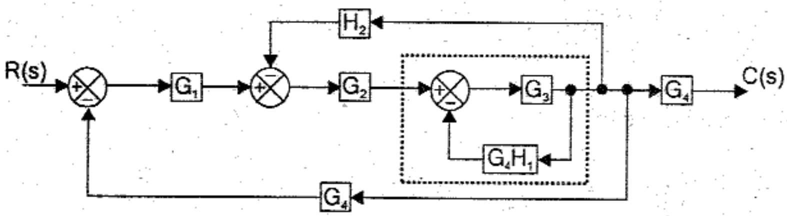
Step 1: Moving the branch point before the block



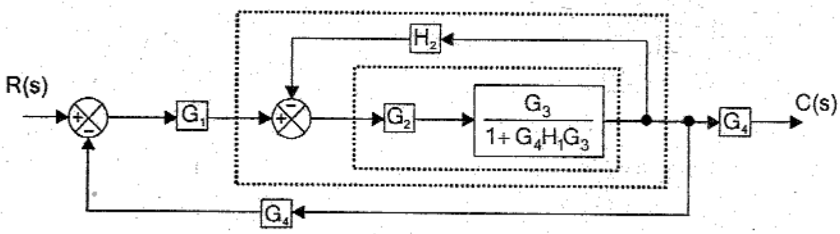
Step 2: Combining the blocks in cascade and rearranging the branch points



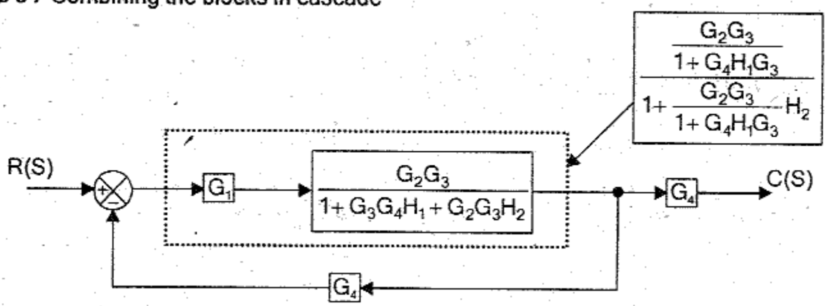
Step 3: Eliminating the feedback path



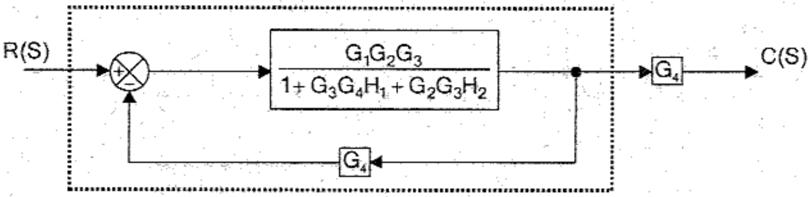
Step 4: Combining the blocks in cascade and eliminating feedback path



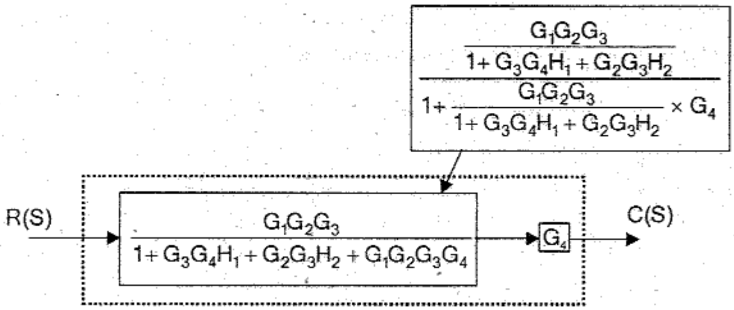
Step 5: Combining the blocks in cascade



Step 6 : Eliminating the feedback path



Step 7 : Combining the blocks in cascade



$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

RESULT

The overall transfer function of the system is given by,

$$\frac{C(s)}{R(s)}=\frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.19

For the system represented by the block diagram shown in fig 1. Evaluate the closed loop transfer function when the input R is (i) at station-I (ii) at station-II.

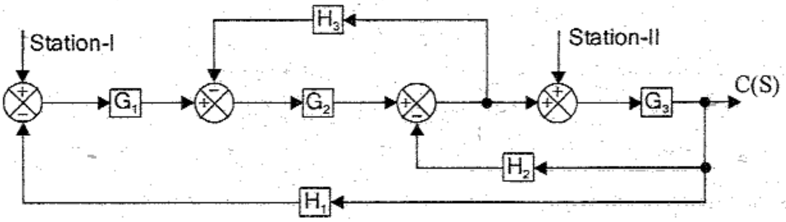


Fig 1.

SOLUTION

(i) Consider the input R is at station-I and so the input at station-II is made zero. Let the output be C_1. Since there is no input at station-II that summing point can be removed and resulting block diagram is shown in fig 2.

Step 1 : Shift the take off point of feedback H_3 beyond G_3 and rearrange the branch points

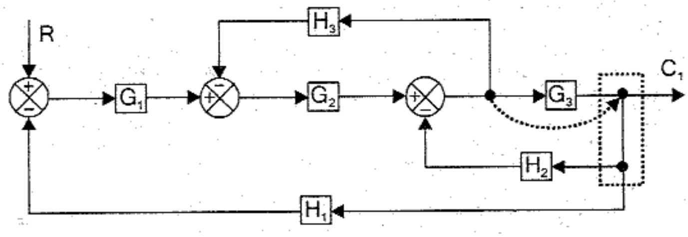
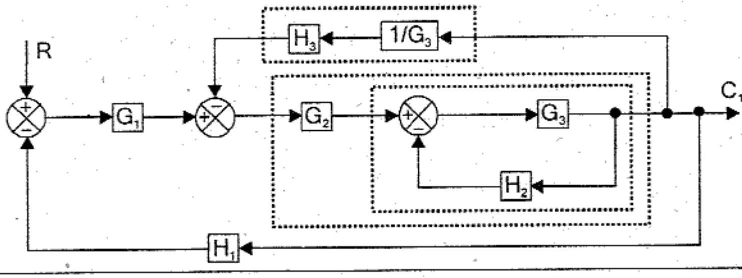
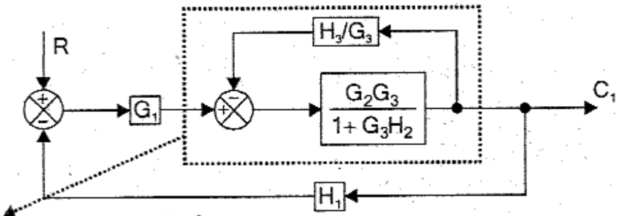


Fig 2.

Step 2 : Eliminating the feedback H_2 and combining blocks in cascade

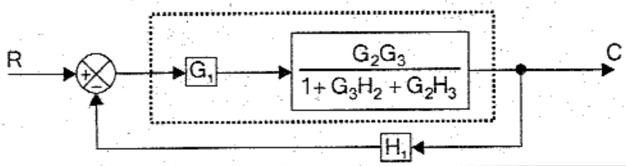


Step 3: Eliminating the feedback path

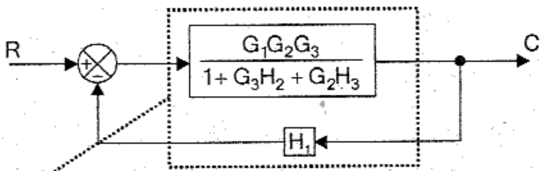


$$\frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2G_3}{1+G_3H_2}\times\dfrac{H_3}{G_3}} = \frac{\dfrac{G_2G_3}{1+G_3H_2}}{1+\dfrac{G_2H_3}{1+G_3H_2}} = \frac{G_2G_3}{1+G_3H_2+G_2H_3}$$

Step 4: Combining the blocks in cascade



Step 5: Eliminating feedback path H_1



$$\frac{\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}}{1+\dfrac{G_1G_2G_3}{1+G_3H_2+G_2H_3}\times H_1} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

$$\therefore \frac{C_1(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

(ii) Consider the input R at station-II, the input at station-I is made zero. Let output be C_2. Since there is no input in station-I that corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_1. The resulting block diagram is shown in fig 3.

Step 1: Combining the blocks in cascade, shifting the summing point of H_2 before G_2 and rearranging the branch points.

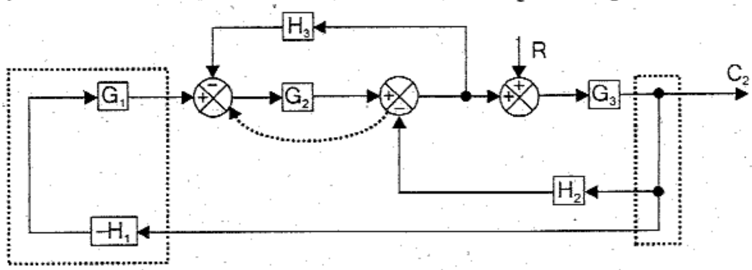
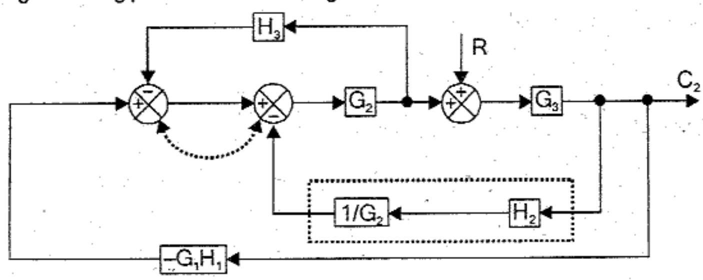
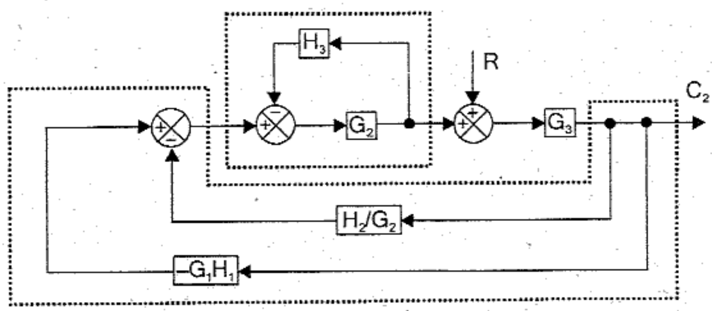


Fig 3.

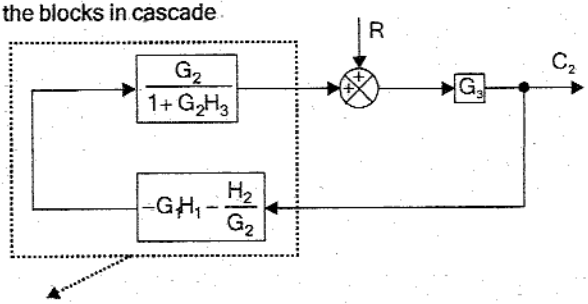
Step 2: Interchanging summing points and combining the blocks in cascade.



Step 3: Combining parallel blocks and eliminating feedback path

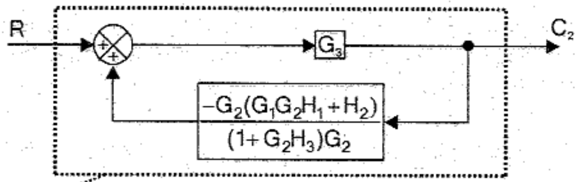


Step 4: Combining the blocks in cascade



$$\left(\frac{G_2}{1+G_2H_3}\right)\times\left(-G_1H_1-\frac{H_2}{G_2}\right)=\left(\frac{G_2}{1+G_2H_3}\right)\times\left(\frac{-G_1H_1G_2-H_2}{G_2}\right)=\frac{-G_2(G_1G_2H_1+H_2)}{(1+G_2H_3)G_2}$$

Step 5: Eliminating the feedback path



$$\frac{G_3}{1-\left(\dfrac{-(G_1G_2H_1+H_2)}{1+G_2H_3}\right)G_3} = \frac{G_3}{\dfrac{1+G_2H_3+G_3(G_1G_2H_1+H_2)}{1+G_2H_3}} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

$$\therefore \quad \frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

RESULT

The transfer function of the system with input at station-I is,

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_2+G_2H_3+G_1G_2G_3H_1}$$

The transfer function of the system with input at station-II is,

$$\frac{C_2}{R} = \frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3(G_1G_2H_1+H_2)}$$

EXAMPLE 1.20

For the system represented by the block diagram shown in the fig 1, determine C_1/R_1 and C_2/R_1.

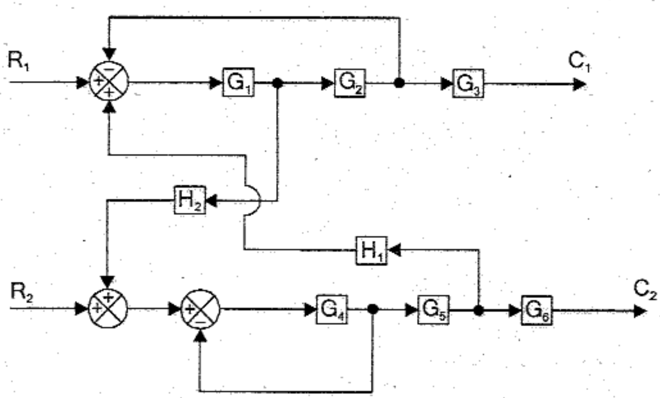


Fig 1

SOLUTION

Case (i) To find $\frac{C_1}{R_1}$

In this case set $R_2 = 0$ and consider only one output C_1. Hence we can remove the summing point which adds R_2 and need not consider G_6, since G_6 is on the open path. The resulting block diagram is shown in fig 2.

Step 1: Eliminating the feedback path

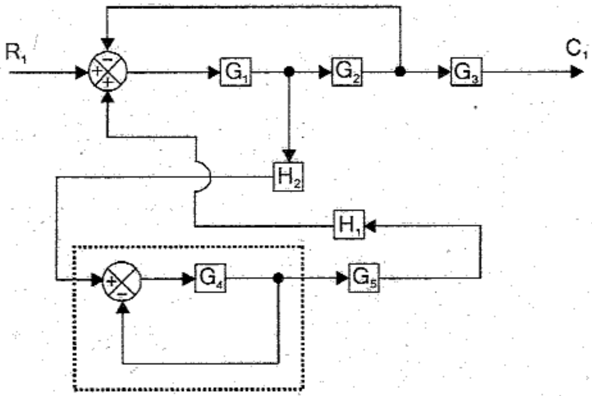
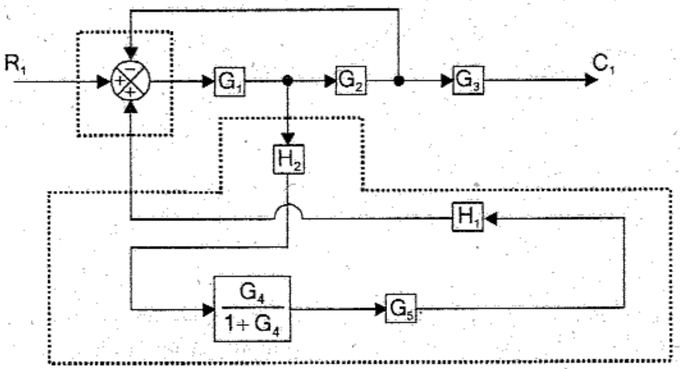
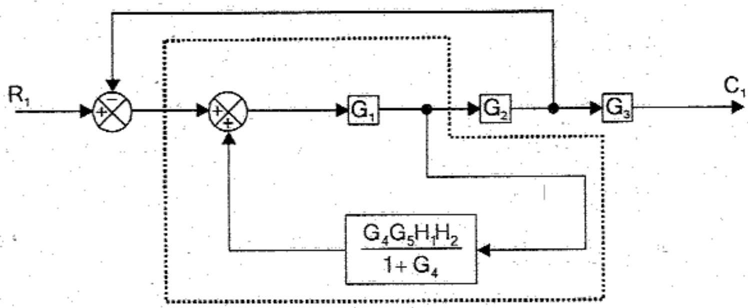


Fig 2.

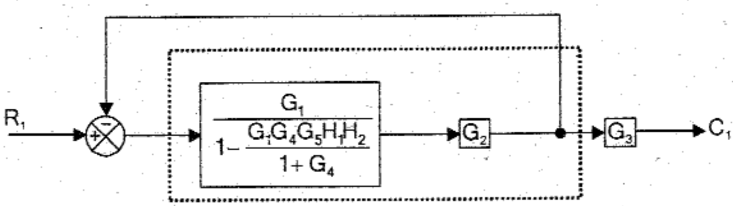
Step 2: Combining the blocks in cascade and splitting the summing point



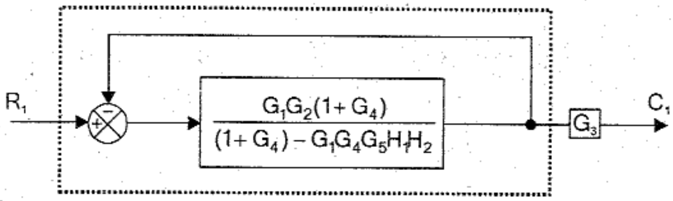
Step 3: Eliminating the feedback path

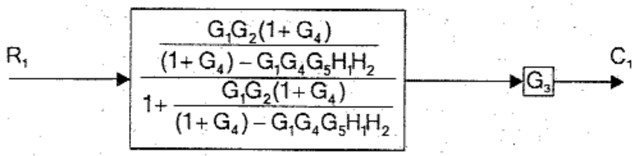


Step 4: Combining the blocks in cascade

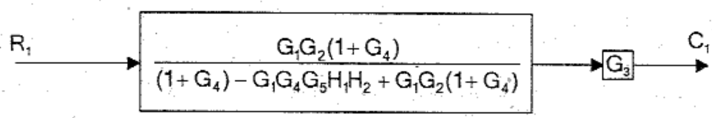


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_1}{R_1}=\frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)\,(1+G_4)-G_1G_4G_5H_1H_2}$$

Case 2 : To find $\frac{C_2}{R_1}$

In this case set $R_2 = 0$ and consider only one output C_2. Hence we can remove the summing point which adds R_2 and need not consider G_3, since G_3 is on the open path. The resulting block diagram is shown in fig 3.

Step 1: Eliminate the feedback path.

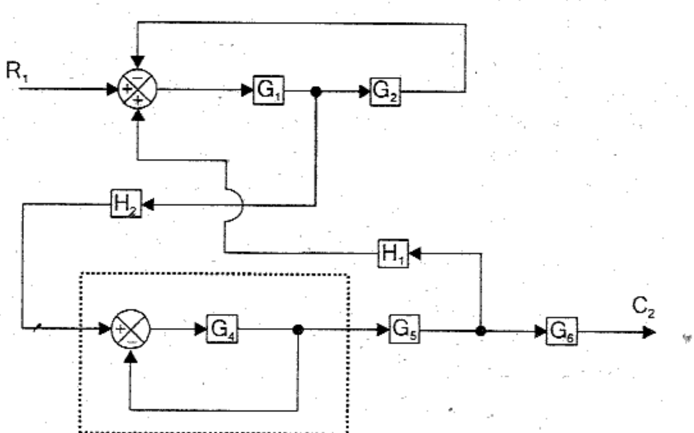
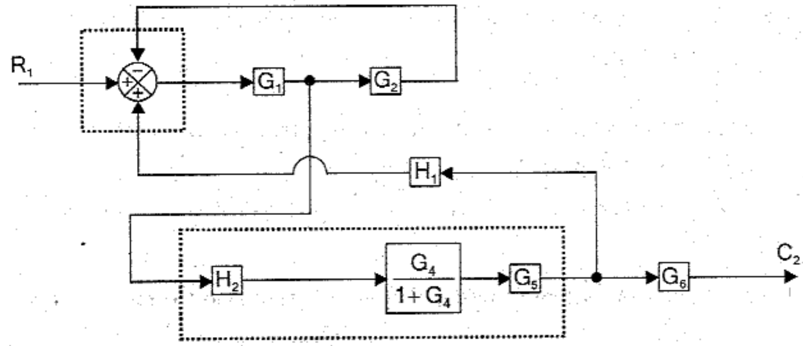
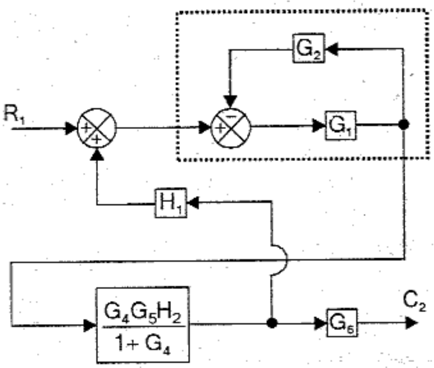


Fig 3.

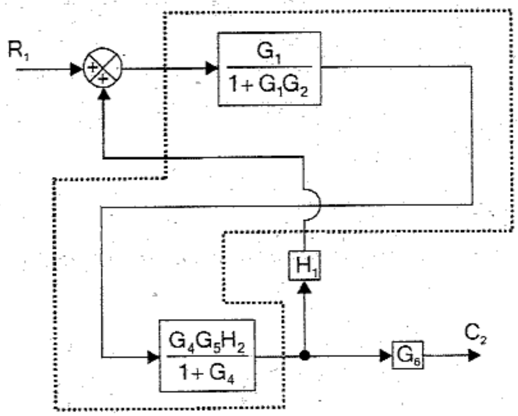
Step 2: Combining blocks in cascade and splitting the summing point



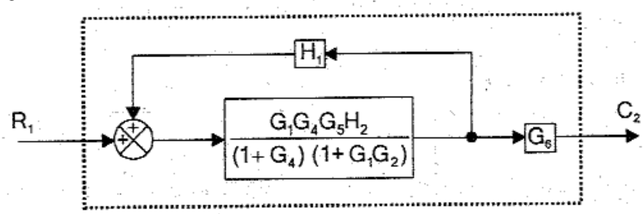
Step 3: Eliminating the feedback path

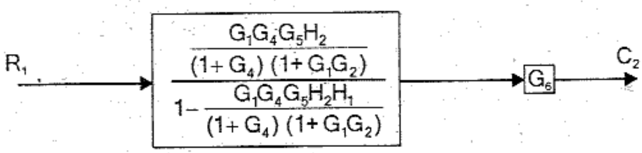


Step 4 : Combining the blocks in cascade

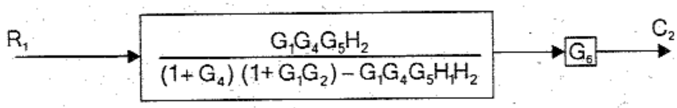


Step 5: Eliminating the feedback path





Step 6: Combining the blocks in cascade



$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)\,(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

RESULT

The transfer function of the system when the input and output are R_1 and C_1 is given by,

$$\frac{C_1}{R_1} = \frac{G_1G_2G_3\,(1+G_4)}{(1+G_1G_2)\,(1+G_4) - G_1G_4G_5H_1H_2}$$

The transfer function of the system when the input and output are R_1 and C_2 is given by,

$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6\,H_2}{(1+G_4)\,(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

EXAMPLE 1.21

Obtain the closed loop transfer function C(s)/R(s) of the system whose block diagram is shown in fig 1.

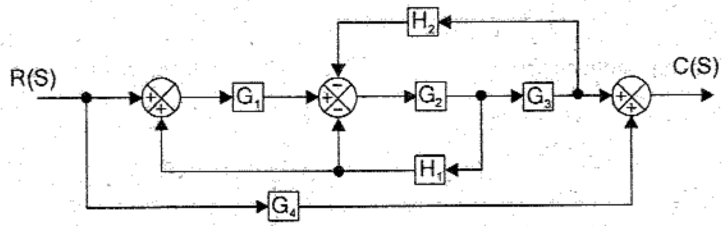
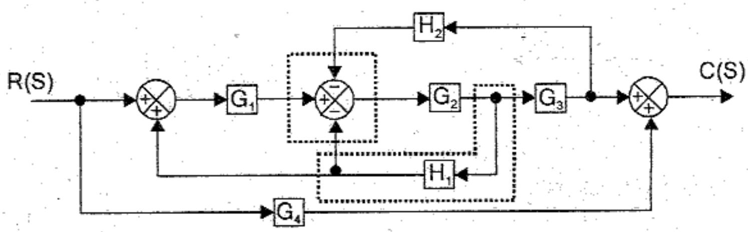


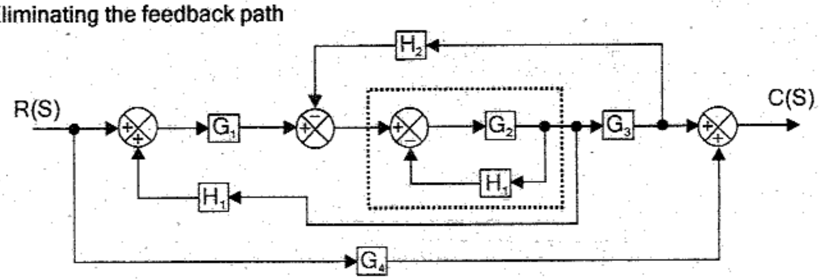
Fig 1.

SOLUTION

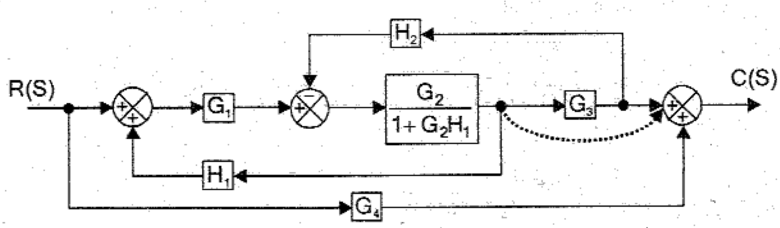
Step 1 : Splitting the summing point and rearranging the branch points



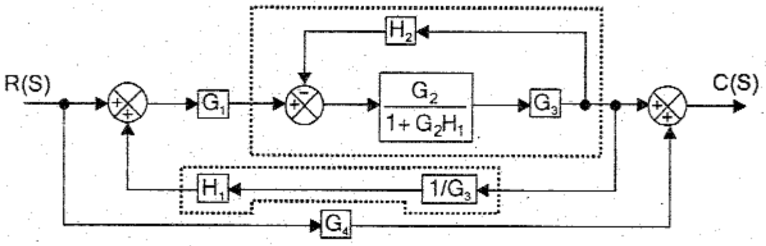
Step 2 : Eliminating the feedback path



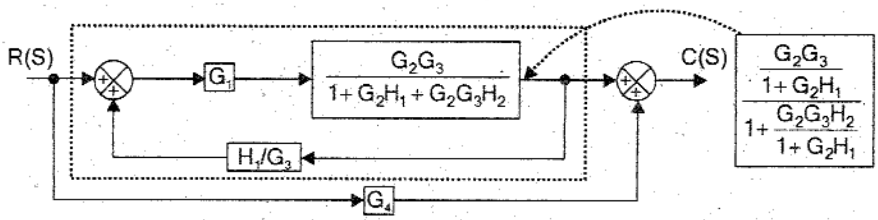
Step 3 : Shifting the branch point after the block.



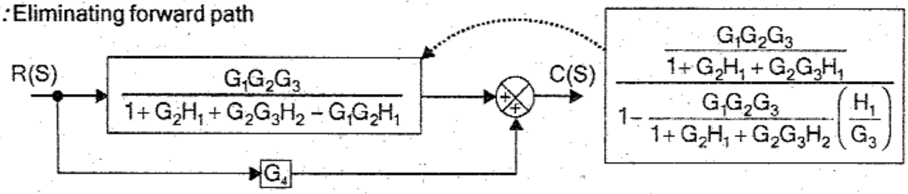
Step 4 : Combining the blocks in cascade and eliminating feedback path



Step 5 : Combining the blocks in cascade and eliminating feedback path



Step 6 : Eliminating forward path



$$\therefore \frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$$

RESULT

The transfer function of the system is $\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1+G_2H_1+G_2G_3H_2-G_1G_2H_1} + G_4$

EXAMPLE 1.22

The block diagram of a closed loop system is shown in fig 1. Using the block diagram reduction technique determine the closed loop transfer function C(s)/R(s).

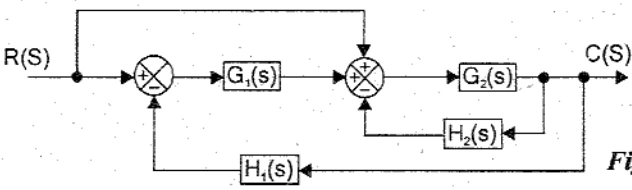
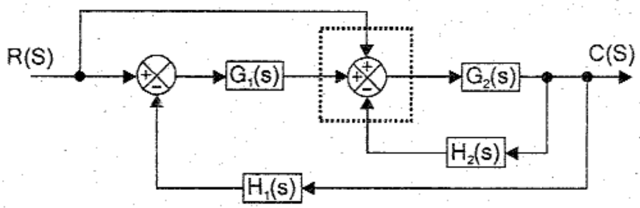


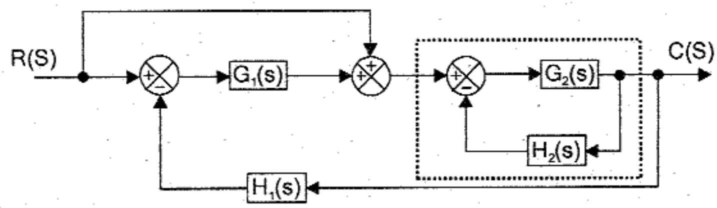
Fig 1.

SOLUTION

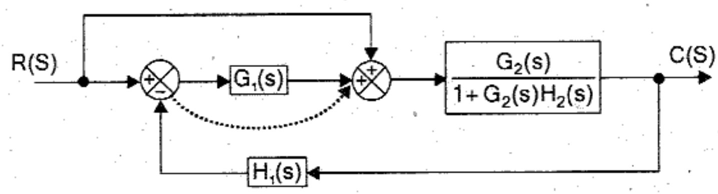
Step 1 : Splitting the summing point.



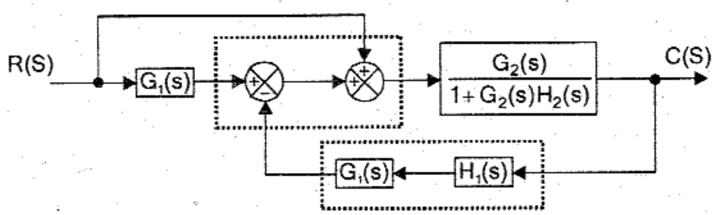
Step 2 : Eliminating the feedback path.



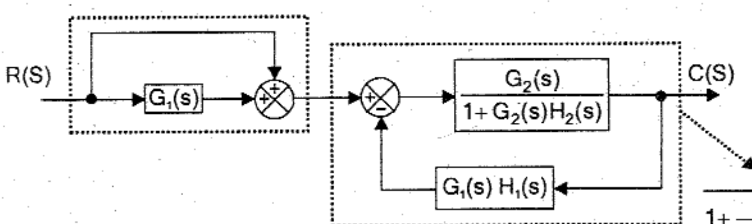
Step 3 : Moving the summing point after the block.



Step 4 : Interchanging the summing points and combining the blocks in cascade

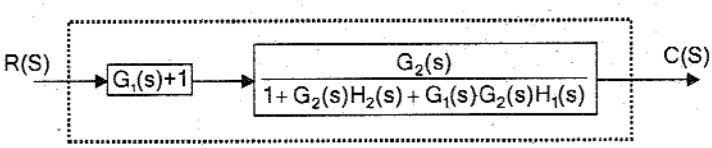


Step 5 : Eliminating the feedback path and feed forward path



$$\frac{\dfrac{G_2(s)}{1+G_2(s)H_2(s)}}{1+\dfrac{G_2(s)}{1+G_2(s)H_2(s)}G_1(s)H_1(s)}$$

Step 6 : Combining the blocks in cascade



$$\therefore \frac{C(s)}{R(s)} = \frac{G_2(s)\left[G_1(s)+1\right]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_2(s)\left[G_1(s)+1\right]}{1+G_2(s)\,H_2(s)+G_1(s)\,G_2(s)\,H_1(s)}$$

EXAMPLE 1.23

Using block diagram reduction technique find the transfer function C(s)/R(s) for the system shown in fig 1.

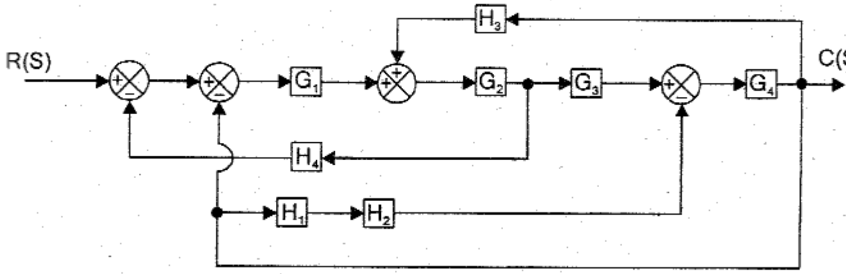
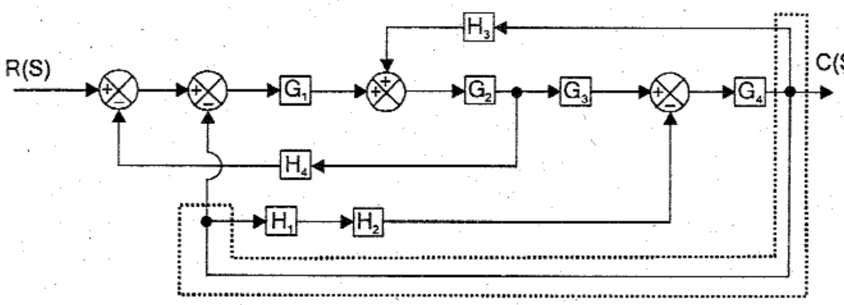


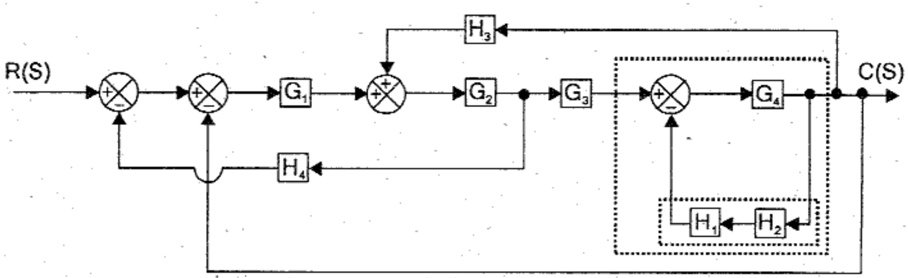
Fig 1.

SOLUTION

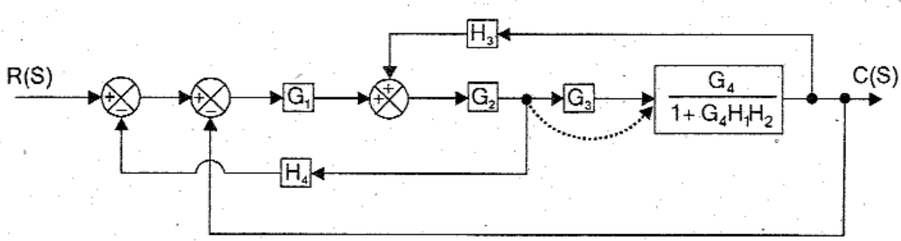
Step 1 : Rearranging the branch points



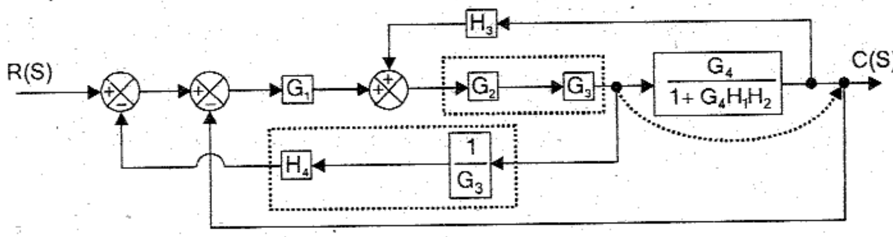
Step 2 : Combining the blocks in cascade and eliminating the feedback path.



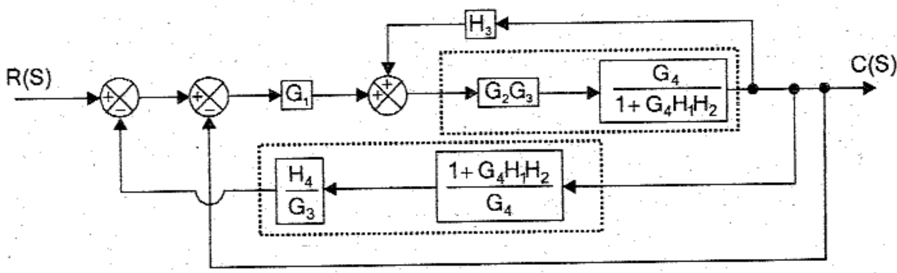
Step 3 : Moving the branch point after the block.



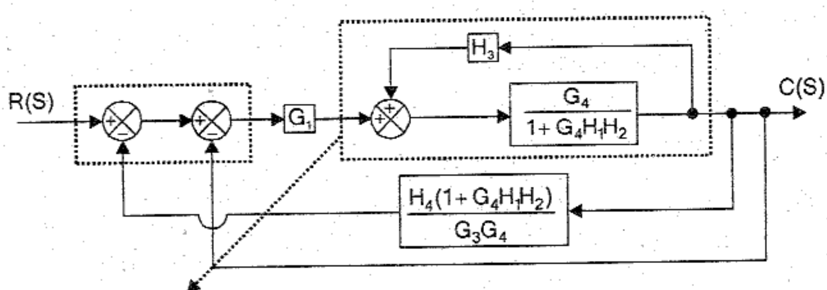
Step 4 : Moving the branch point and combining the blocks in cascade.



Step 5 : Combining the blocks in cascade

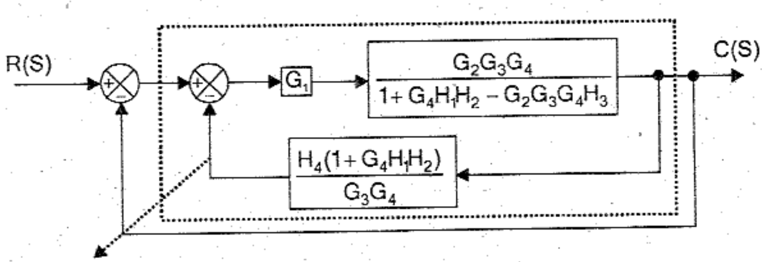


Step 6 : Eliminating feedback path and interchanging the summing points.



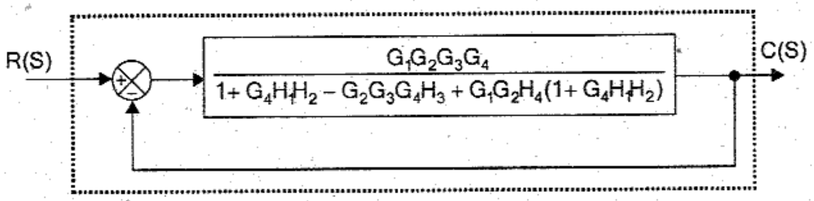
$$\frac{\dfrac{G_2G_3G_4}{1+G_4H_1H_2}}{1-\dfrac{G_2G_3G_4H_3}{1+G_4H_1H_2}} = \frac{G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}$$

Step 7 : Combining the blocks in cascade and eliminating the feedback path



$$\frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{G_3G_4}\right)} = \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}$$

Step 8 : Eliminating the unity feedback path.



$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$= \frac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)+G_1G_2G_3G_4}$$

$$= \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

RESULT

The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}$$

1.12 BLOCK DIAGRAM REDUCTION USING MATLAB

TRANSFER FUNCTION OF A SYSTEM

Let, G(s) be the transfer function of a system. When the transfer function is a rational function of "s", then using MATLAB the transfer function can be obtained from the coefficients of the numerator and denominator polynomials as shown below. Let, the general form of G(s) be as shown below.

$$G(s) = \frac{b_0s^M + b_1s^{M-1} + b_2s^{M-2} + \dots\dots\dots + b_{M-1}s + b_M}{a_0s^N + a_1s^{N-1} + a_2s^{N-2} + \dots\dots\dots + a_{N-1}s + a_N}$$

First, the coefficients of the numerator and denominator polynomials are declared as two arrays as shown below.

```
num_cof  =  [b0  b1  b2  ........  bM];
den_cof  =  [a0  a1  a2  ........  aN];
```

Next, the transfer can be obtained using the following commands of MATLAB.

```
G  =  tf('s');
G  =  ([num_cof],  [den_cof])
```

TRANSFER FUNCTION OF CASCADE / PARALLEL / FEEDBACK SYSTEM

Consider two systems with transfer functions $G_1(s)$ and $G_2(s)$. Let the two transfer functions be rational function of "s" as shown below.

$$G_1(s) = \frac{b_0s^M + b_1s^{M-1} + b_2s^{M-2} + \dots\dots\dots + b_{M-1}s + b_M}{a_0s^N + a_1s^{N-1} + a_2s^{N-2} + \dots\dots\dots + a_{N-1}s + a_N}$$

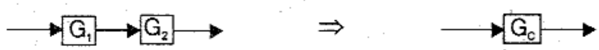
$$G_2(s) = \frac{d_0 s^M + d_1 s^{M-1} + d_2 s^{M-2} + \ldots\ldots\ldots + d_{M-1} s + d_M}{c_0 s^N + c_1 s^{N-1} + c_2 s^{N-2} + \ldots\ldots\ldots + c_{N-1} s + c_N}$$

When the two systems are connected as cascade / parallel / feedback system, then the over transfer function of cascaded system / parallel system / feedback system can be obtained using MATLA

In order to obtain the overall transfer function, first the coefficients of the numerator and denomina polynomials of $G_1(s)$ and $G_2(s)$ are declared as arrays as shown below.

```
num_cof1  =  [b0  b1  b2  ........  bM];
den_cof1  =  [a0  a1  a2  ........  aN];
num_cof2  =  [d0  d1  d2  ........  dM];
den_cof2  =  [c0  c1  c2  ........  cN];
```

When the two systems are connected in cascade as shown below, then the overall transfer functio $G_C(s)$ of the cascaded system can be obtained using the following commands of MATLAB.

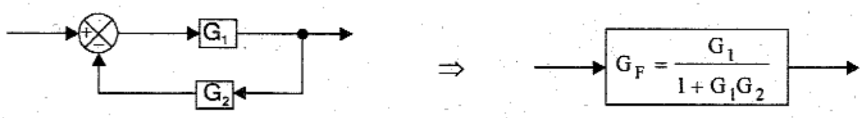



```
GC = tf('s');
[num_cofC, den_cofC] = series(num_cof1, den_cof1, num_cof2, den_cof2);
GC = ([num_cofC], [den_cofC])
```

When the two systems are connected in parallel as shown below, then the overall transfer functio $G_P(s)$ of parallel system can be obtained using the following commands of MATLAB.


```
GP = tf('s');
[num_cofP, den_cofP]=parallel(num_cof1, den_cof1, num_cof2, den_cof2);
GP = ([num_cofP], [den_cofP])
```

When the two systems are connected in feedback as shown below, then the overall trans function $G_F(s)$ of feedback system can be obtained using the following commands of MATLAB.




```
GF = tf('s');
[num_cofF, den_cofF] = feedback(num_cof1, den_cof1, num_cof2, den_cof2)
GF = ([num_cofF], [den_cofF])
```

PROGRAM 1.1

Consider the transfer functions of the two systems given below.

$G_1(s)=8/(s^2+2s+9)$ and $G_2(s)=4/(s+6)$

Write a MATLAB program to find the overall transfer function if the two systems are connected as cascade system, parallel system and feedback system.

```
clc
clear all
G1=tf('s'); G2=tf('s'); GC=tf('s');GP=tf('s');GF=tf('s');
num_cof1=[0 0 8];
den_cof1=[1 2 9];
disp('System1');
G1=tf([num_cof1], [den_cof1])
num_cof2=[0 4];
den_cof2=[1 6];
disp('System2');
G2=tf([num_cof2], [den_cof2])
[num_cofC,den_cofC]=series(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Cascade system');
GC=tf([num_cofC], [den_cofC])
[num_cofP,den_cofP]=parallel(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Parallel system');
GP=tf([num_cofP], [den_cofP])
[num_cofF,den_cofF]=feedback(num_cof1,den_cof1,num_cof2,den_cof2);
disp('Feedback system');
GF=tf([num_cofF], [den_cofF])
```

OUTPUT

```
System1
Transfer function:
          8
     -------------
     s^2 + 2 s + 9

System2
Transfer function:
       4
     -----
     s + 6

Cascade system
Transfer function:
                32
     ----------------------
     s^3 + 8 s^2 + 21 s + 54

Parallel system
Transfer function:
        4 s^2 + 16 s + 84
     ----------------------
     s^3 + 8 s^2 + 21 s + 54

Feedback system
Transfer function:
            8 s + 48
     ----------------------
     s^3 + 8 s^2 + 21 s + 86
```

1.13 SIGNAL FLOW GRAPH

The signal flow graph is used to represent the control system graphically and it was developed by **S.J. Mason.**

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

It should be noted that the signal flow graph approach and the block diagram approach yield the same information. The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily. This method is simpler than the tedious block diagram reduction techniques.

The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals. A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable and each branch connected between two nodes acts as a signal multiplier. Each branch has a gain or transmittance. When the signal pass through a branch, it gets multiplied by the gain of the branch.

In a signal flow graph, the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch and the gain (multiplication factor) is indicated along the branch.

EXPLANATION OF TERMS USED IN SIGNAL FLOW GRAPH

Node : A node is a point representing a variable or signal.

Branch : A branch is directed line segment joining two nodes. The arrow on the branch indicates the direction of signal flow and the gain of a branch is the transmittance.

Transmittance : The gain acquired by the signal when it travels from one node to another is called transmittance. The transmittance can be real or complex.

Input node (Source) : It is a node that has only outgoing branches.

Output node (Sink) : It is a node that has only incoming branches.

Mixed node : It is a node that has both incoming and outgoing branches.

Path : A path is a traversal of connected branches in the direction of the branch arrows. The path should not cross a node more than once.

Open path : A open path starts at a node and ends at another node.

Closed path : Closed path starts and ends at same node.

Forward path : It is a path from an input node to an output node that does not cross any node more than once.

***Forward path* gain** : It is the product of the branch transmittances (gains) of a forward path.

Individual loop : It is a closed path starting from a node and after passing through a certain part of a graph arrives at same node without crossing any node more than once.

Loop gain : It is the product of the branch transmittances (gains) of a loop.

Non-touching Loops : If the loops does not have a common node then they are said to be non-touching loops.

ROPERTIES OF SIGNAL FLOW GRAPH

The basic properties of signal flow graph are the following :

(i) The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect relationship.

(ii) Signal flow graph is applicable to linear systems only.

(iii) A node in the signal flow graph represents the variable or signal.

(iv) A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.

(v) A mixed node which has both incoming and outgoing signals can be treated as an output node by adding an outgoing branch of unity transmittance.

(vi) A branch indicates functional dependence of one signal on the other.

(vii) The signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

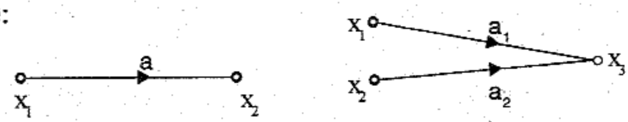
(viii) The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

GNAL FLOW GRAPH ALGEBRA

Signal flow graph for a system can be reduced to obtain the transfer function of the system using e following rules. The guideline in developing the rules for signal flow graph algebra is that the signal at node is given by sum of all incoming signals.

Rule 1 **:** Incoming signal to a node through a branch is given by the product of a signal at previous node and the gain of the branch.

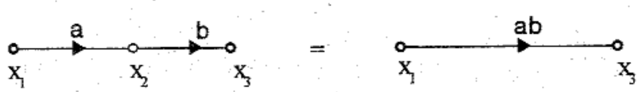
Example:



$x_2 = ax_1$ $\qquad$ $x_3 = a_1x_1 + a_2x_2$

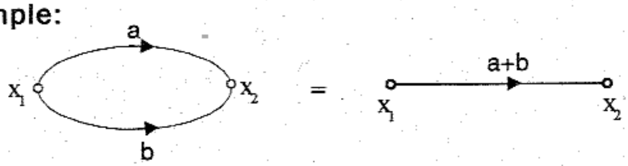
Rule 2 **:** Cascaded branches can be combined to give a single branch whose transmittance is equal to the product of individual branch transmittance.

Example:



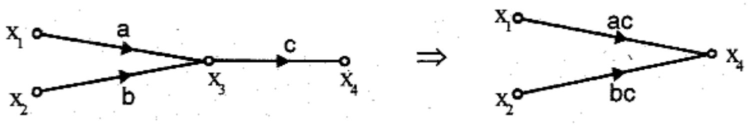
Rule 3 **:** Parallel branches may be represented by single branch whose transmittance is the sum of individual branch transmittances.

Example:



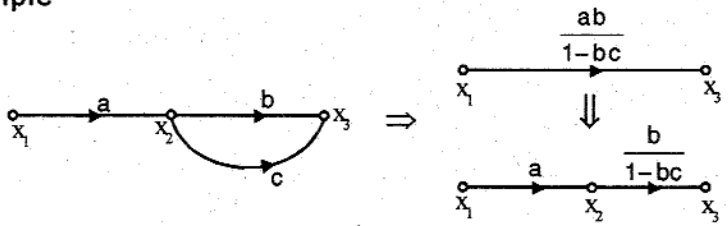
Rule 4 : A mixed node can be eliminated by multiplying the transmittance of outgoing branch (from the mixed node) to the transmittance of all incoming branches to the mixed node.

Example



Rule 5 : A loop may be eliminated by writing equations at the input and output node and rearranging the equations to find the ratio of output to input. This ratio gives the gain of resultant branch.

Example



***Proof*:**

$x_2 = ax_1 + cx_3$; $x_3 = bx_2$

Put, $x_2 = ax_1 + cx_3$ in the equation for x_3.

$\therefore x_3 = b\,(ax_1 + cx_3) \Rightarrow x_3 = abx_1 + bcx_3 \Rightarrow x_3 - bc\,x_3 = ab\,x_1 \Rightarrow x_3(1-bc) = ab\,x_1$

$\therefore \dfrac{x_3}{x_1} = \dfrac{ab}{1-bc}$

SIGNAL FLOW GRAPH REDUCTION

The signal flow graph of a system can be reduced either by using the rules of a signal flow graph algebra or by using Mason's gain formula.

For signal flow graph reduction using the rules of signal flow graph, write equations at every node and then rearrange these equations to get the ratio of output and input (transfer function).

The signal flow graph reduction by above method will be time consuming and tedious. **S.J.Mason** has developed a simple procedure to determine the transfer function of the system represented as a signal flow graph. He has developed a formula called by his name **Mason's gain formula** which can be directly used to find the transfer function of the system.

MASON'S GAIN FORMULA

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, R(s) = Input to the system

C(s) = Output of the system

Now, Transfer function of the system, $T(s) = \dfrac{C(s)}{R(s)}$ .....(1.34)

Mason's gain formula states the overall gain of the system [transfer function] as follows,

Overall gain, $T = \dfrac{1}{\Delta} \sum_K P_K \Delta_K$ .....(1.35) where, $T = T(s)$ = Transfer function of the system

P_K = Forward path gain of K^{th} forward path

K = Number of forward paths in the signal flow graph

$$\Delta = 1 - (\text{Sum of individual loop gains}) + \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of two non - touching loops}\end{pmatrix} - \begin{pmatrix}\text{Sum of gain products of all possible} \\ \text{combinations of three non - touching loops}\end{pmatrix} + \ldots\ldots$$

Δ_K = Δ for that part of the graph which is not touching K^{th} forward path

CONSTRUCTING SIGNAL FLOW GRAPH FOR CONTROL SYSTEMS

A control system can be represented diagrammatically by signal flow graph. The differential equations governing the system are used to construct the signal flow graph.The following procedure can be used to construct the signal flow graph of a system.

1. Take Laplace transform of the differential equations governing the system in order to convert them to algebraic equations in s-domain.
2. The constants and variables of the s-domain equations are identified.
3. From the working knowledge of the system, the variables are identified as input, output and intermediate variables.
4. For each variable a node is assigned in signal flow graph and constants are assigned as the gain or transmittance of the branches connecting the nodes.
5. For each equation a signal flow graph is drawn and then they are interconnected to give overall signal flow graph of the system.

PROCEDURE FOR CONVERTING BLOCK DIAGRAM TO SIGNAL FLOW GRAPH

The signal flow graph and block diagram of a system provides the same information but there is no standard procedure for reducing the block diagram to find the transfer function of the system. Also the block diagram reduction technique will be tedious and it is difficult to choose the rule to be applied for simplification. Hence it will be easier if the block diagram is converted to signal flow graph and **Mason's gain formula** is applied to find the transfer function. The following procedure can be used to convert block diagram to signal flow graph.

1. Assume nodes at input, output, at every summing point, at every branch point and in between cascaded blocks.
2. Draw the nodes separately as small circles and number the circles in the order 1, 2, 3, 4, ... etc.
3. From the block diagram find the gain between each node in the main forward path and connect all the corresponding circles by straight line and mark the gain between the nodes.
4. Draw the feed forward paths between various nodes and mark the gain of feed forward path along with sign.
5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with sign.

EXAMPLE 1.24

Construct a signal flow graph for armature controlled dc motor.

SOLUTION

The differential equations governing the armature controlled dc motor are (refer section 1.7).

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \,; \qquad T = K_t i_a \,; \qquad T = J\,\frac{d\omega}{dt} + B\omega \,; \qquad e_b = K_b\,\omega \,; \qquad \omega = d\theta/dt$$

On taking Laplace transform of above equations we get,

$$V_a(s) = I_a(s)\,R_a + L_a\,s\,I_a(s) + E_b(s) \qquad .....(1)$$

$$T(s) = K_t I_a(s) \qquad .....(2)$$

$$T(s) = J\,s\,\omega(s) + B\,\omega(s) \qquad .....(3)$$

$$E_b(s) = K_b\,\omega(s) \qquad .....(4)$$

$$\omega(s) = s\,\theta(s) \qquad .....(5)$$

The input and output variables of armature controlled dc motor are armature voltage $V_a(s)$ and angular displacement $\theta(s)$ respectively. The variables $I_a(s)$, $T(s)$, $E_b(s)$ and $\omega(s)$ are intermediate variables.

The equations (1) to (5) are rearranged & individual signal flow graph are shown in fig 1 to fig 5.

$$V_a(s) - E_b(s) = I_a(s)\,[R_a + s\,L_a]$$

$$\therefore I_a(s) = \frac{1}{R_a + s\,L_a}\,[V_a(s) - E_b(s)]$$

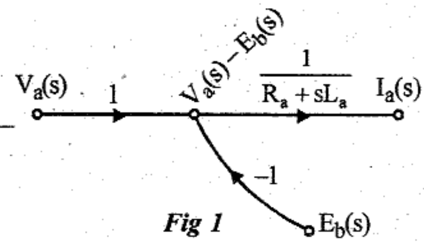


Fig 1

$$T(s) = K_t I_a(s)$$

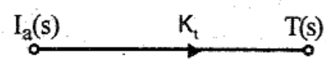


Fig 1

$$T(s) = \omega(s)\,[Js + B]$$

$$\therefore \omega(s) = \frac{1}{Js + B}\,T(s)$$

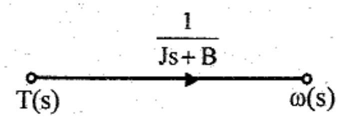


Fig 3

$$E_b(s) = K_b\,\omega(s)$$

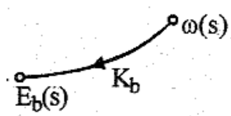


Fig 4

$$\omega(s) = s\theta(s)$$

$$\therefore \theta(s) = \frac{1}{s}\omega(s)$$

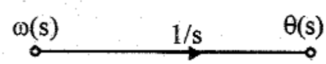


Fig 5

The overall signal flow graph of armature controlled dc motor is obtained by interconnecting the individual signal flow graphs shown in fig 1 to fig 5. The overall signal flow graph is shown in fig 6.

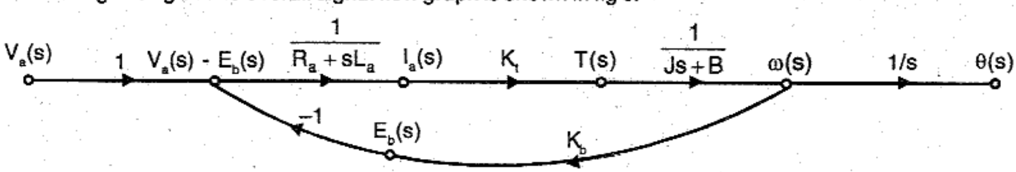


***Fig 6 :** Signal flow graph of armature controlled dc motor.*

EXAMPLE 1.25

Find the overall transfer function of the system whose signal flow graph is shown in fig 1.

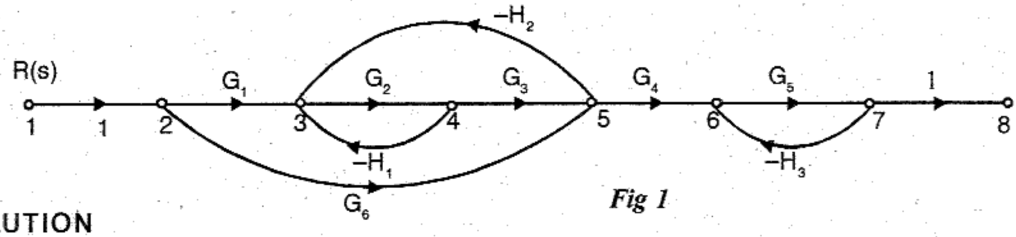


Fig 1

SOLUTION

Forward Path Gains

There are two forward paths. $\therefore K = 2$

Let forward path gains be P_1 and P_2.

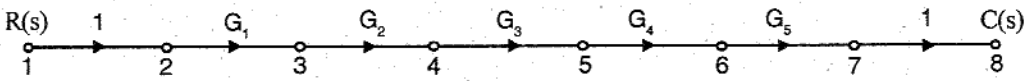


***Fig 2 :** Forward path-1.*

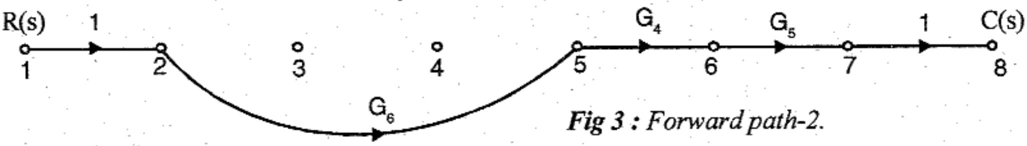


***Fig 3 :** Forward path-2.*

Gain of forward path-1, $P_1 = G_1\ G_2 G_3 G_4 G_5$

Gain of forward path-2, $P_2 = G_4 G_5 G_6$

Individual Loop Gain

There are three individual loops. Let individual loop gains be P_{11}, P_{21} and P_{31}.

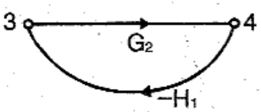


***Fig 4 :** Loop-1.*

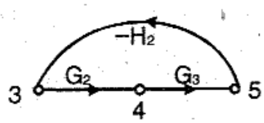


***Fig 5 :** Loop-2.*

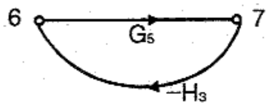


***Fig 6 :** Loop-3.*

Loop gain of individual loop-1, $P_{11} = -G_2 H_1$

Loop gain of individual loop-2, $P_{21} = -G_2 G_3 H_2$

Loop gain of individual loop-3, $P_{31} = -G_5 H_3$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops. Let the gain products of two non touching loops be P_{12} and P_{22}.

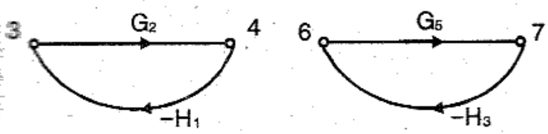


***Fig 7 :** First combination of 2 non-touching loops.*

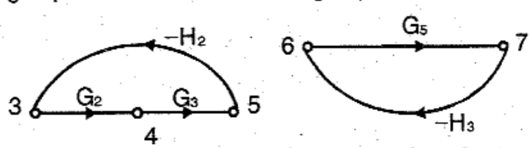


***Fig 8 :** Second combination of 2 non-touching loops.*

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non touching loops}\end{array}\right\} P_{12} = P_{11}P_{31} = (-G_2H_1)\,(-G_5H_3) \ = \ G_2G_5H_1H_3$$

$$\left.\begin{array}{l}\text{Gain product of second combination}\\ \text{of two non touching loops}\end{array}\right\} P_{22} = P_{21}P_{31} = (-G_2G_3H_2)\,(-G_5H_3) \ = \ G_2G_3G_5H_2H_3$$

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

$$= 1 - (-G_2H_1 - G_2G_3H_2 - G_5H_3) + (G_2G_5H_1H_3 + G_2G_3G_5H_2H_3)$$

$$= 1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3$$

$\Delta_1 = 1$, Since there is no part of graph which is not touching with first forward path.

The part of the graph which is non touching with second forward path is shown in fig 9.

$$\Delta_2 = 1 - P_{11} = 1 - (-G_2H_1) = 1 + G_2H_1$$

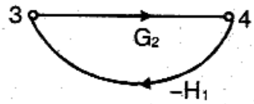


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2) \qquad \text{(Number of forward paths is 2 and so K = 2)}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6\,(1 + G_2H_1)}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6 + G_2G_4G_5G_6H_1}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

$$= \frac{G_2G_4G_5\left[G_1G_3 + G_6 / G_2 + G_6H_1\right]}{1 + G_2H_1 + G_2G_3H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}$$

EXAMPLE 1.26

Find the overall gain of the system whose signal flow graph is shown in fig 1.

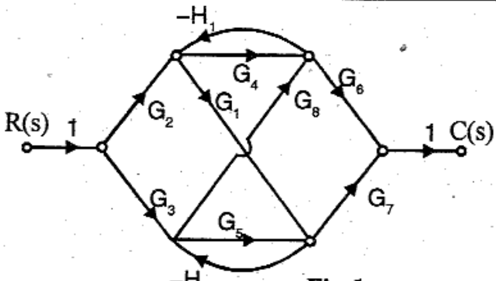


Fig 1

SOLUTION

Let us number the nodes as shown in fig 2.

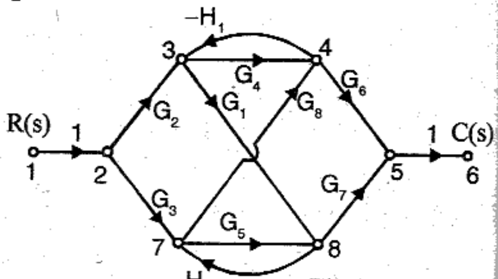


Fig 2

I. Forward Path Gains

There are six forward paths. $\therefore K = 6$

Let the forward path gains be P_1, P_2, P_3, P_4, P_5 and P_6.

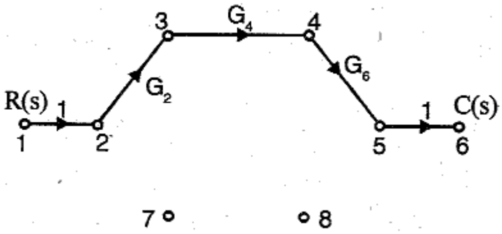


Fig 3 : Forward path-1.

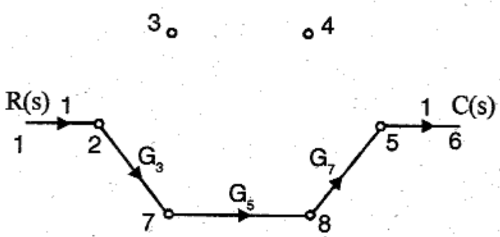


Fig 4 : Forward path-2.

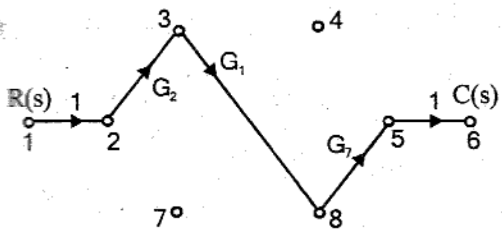


Fig 5 : Forward path-3

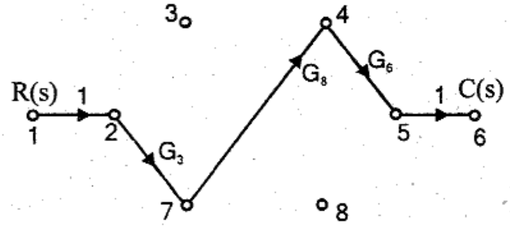


Fig 6 : Forward path-4

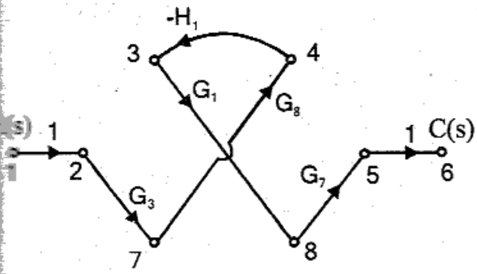


Fig 7 : Forward path-5

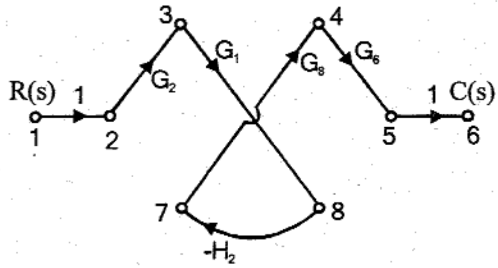


Fig 8 : Forward path-6

Gain of forward path-1, $P_1 = G_2\,G_4\,G_6$

Gain of forward path-2, $P_2 = G_3\,G_5\,G_7$

Gain of forward path-3, $P_3 = G_1\,G_2\,G_7$

Gain of forward path-4, $P_4 = G_3\,G_8\,G_6$

Gain of forward path-5, $P_5 = -G_1G_3G_7G_8H_1$

Gain of forward path-6, $P_6 = -G_1G_2G_6G_8H_2$

Individual Loop Gain

There are three individual loops.

Let individual loop gains be P_{11}, P_{21} and P_{31}.

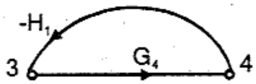


Fig 9 : Loop-1

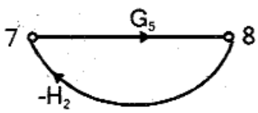


Fig 10 : Loop-2

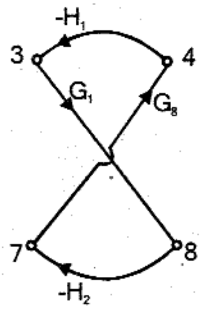


Fig 11 : Loop-3

Loop gain of individual loop-1, $P_{11} = -G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_5H_2$

Loop gain of individual loop-3, $P_{31} = G_1G_8H_1H_2$

Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops. Let gain product of two non-touching loops be P_{12}.

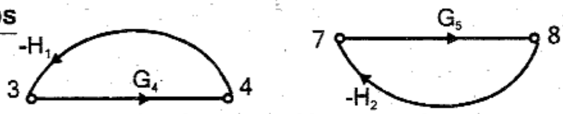


Fig 12 : Combination of 2 non-touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non-touching loops}\end{array}\right\} P_{12} = P_{11}P_{21} = (-G_4H_1)\,(-G_5H_2) = G_4G_5H_1H_2$$

Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12} = 1 - (-G_4H_1 - G_5H_2 + G_1G_8H_1H_2) + G_4G_5H_1H_2$$

$$= 1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2$$

The part of the graph non-touching forward path - 1 is shown in fig 13.

$$\therefore \Delta_1 = 1 - (-G_5H_2) = 1 + G_5H_2$$

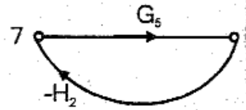


Fig 13

The part of the graph non-touching forward path -2 is shown in fig 14.

$$\therefore \Delta_2 = 1 - (-G_4H_1) = 1 + G_4H_1$$

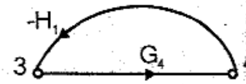


Fig 14

There is no part of the graph which is non-touching with forward paths 3, 4, 5 and 6.

$$\therefore \Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1$$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\left(\sum_K P_K \Delta_K\right) \qquad \text{(Number of forward paths is six and so K = 6)}$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6)$$

$$= \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_6G_8 - G_1G_3G_7G_8H_1 - G_1G_2G_6G_8H_2}{1 + G_4H_1 + G_5H_2 - G_1G_8H_1H_2 + G_4G_5H_1H_2}$$

EXAMPLE 1.27

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

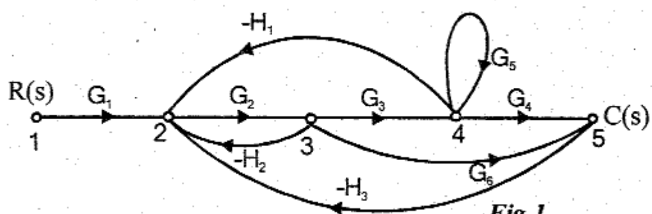


Fig 1

SOLUTION

I. Forward Path Gains

There are two forward paths. $\therefore K = 2$. Let the forward path gains be P_1 and P_2.

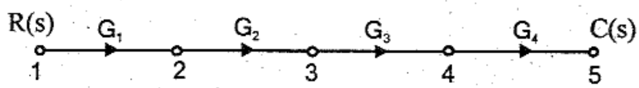


Fig 2 : Forward path-1

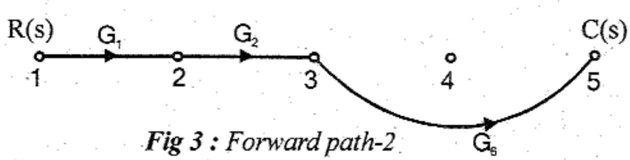


Fig 3 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

Gain of forward path-2, $P_2 = G_1G_2G_6$

ndividual Loop Gain

There are five individual loops. Let the individual loop gains be p_{11}, p_{21}, p_{31}, p_{41} and p_{51}.

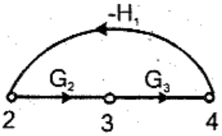


Fig 4 : loop-1

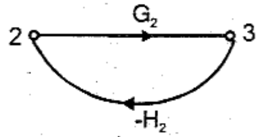


Fig 5 : loop-2

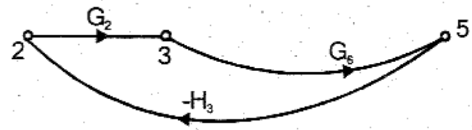


Fig 6 : loop-3

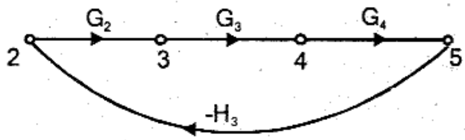


Fig 7 : loop-4

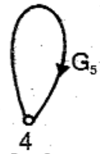


Fig 8 : loop-5

Loop gain of individual loop-1, $P_{11} = -G_2G_3H_1$

Loop gain of individual loop-2, $P_{21} = -H_2G_2$

Loop gain of individual loop-3, $P_{31} = -G_2G_6H_3$

Loop gain of individual loop-4, $P_{41} = -G_2G_3G_4H_3$

Loop gain of individual loop-5, $P_{51} = G_5$

Gain Products of Two Non-touching Loops

There are two combinations of two non-touching loops.
Let the gain products of two non-touching loops be P_{12} and P_{22}.

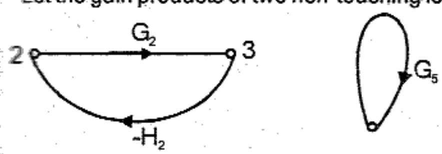


Fig 9 : First combination of two non-touching loops

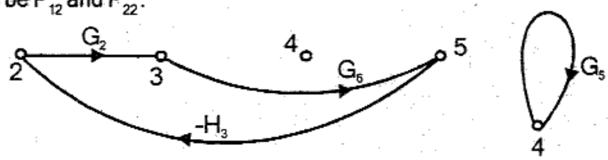


Fig 10 : Second combination of two non-touching loops

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non touching loops}\end{array}\right\} P_{12} = P_{21}P_{51} = (-G_2H_2)(G_5) = G_2G_5H_2$$

$$\left.\begin{array}{l}\text{Gain product of second combination}\\ \text{of two non touching loops}\end{array}\right\} P_{22} = P_{31}P_{51} = (-G_2G_6H_3)(G_5) = -G_2G_5G_6H_3$$

Calculation of Δ and Δ_K

$$\begin{aligned}\Delta &= 1-(P_{11}+P_{21}+P_{31}+P_{41}+P_{51})+(P_{12}+P_{22})\\ &= 1-(-G_2G_3H_1 - H_2G_2 - G_2G_3G_4H_3 + G_5 - G_2G_6H_3)\\ &\quad +(-G_2H_2G_5 - G_2G_5G_6H_3)\end{aligned}$$

Since there is no part of graph which is not touching forward path-1, $\Delta_1 = 1$.

The part of graph which is not touching forward path-2 is shown in fig 11.

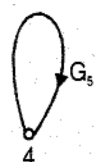


Fig 11

$\therefore \Delta_2 = 1 - G_5$

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K \text{ (Number of forward path is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}\left[P_1\Delta_1 + P_2\Delta_2\right] = \frac{1}{\Delta}\left[G_1G_2G_3G_4 \times 1 + G_1G_2G_6(1-G_5)\right]$$

$$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + G_2G_3H_1 + H_2G_2 + G_2G_3G_4H_3 - G_5 + G_2G_6H_3 - G_2H_2G_5 - G_2G_5G_6H_3}$$

EXAMPLE 1.28

Find the overall gain C(s)/R(s) for the signal flow graph shown in fig 1.

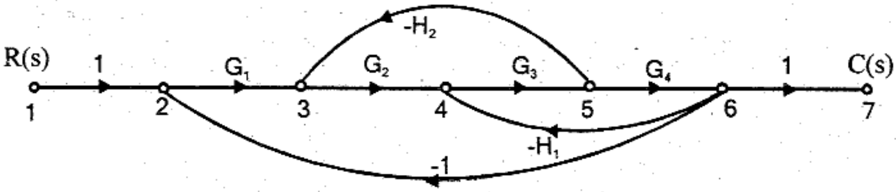


Fig 1

SOLUTION

I. Forward Path Gains

There is only one forward path. $\therefore K = 1$.

Let the forward path gain be P_1.

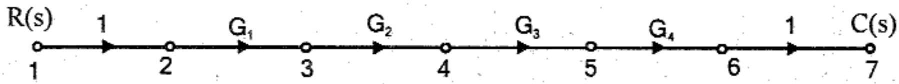


Fig 1 : Forward path-1

Gain of forward path-1, $P_1 = G_1G_2G_3G_4$

II. Individual Loop Gain

There are three individual loops. Let the loop gains be P_{11}, P_{21}, P_{31}.

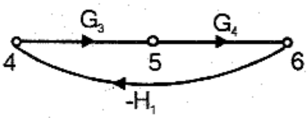


Fig 3 : loop-1

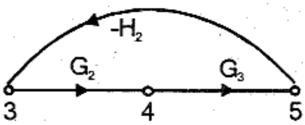


Fig 4 : loop-2

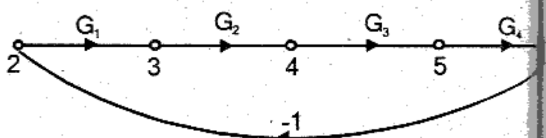


Fig 5 : loop-3

Loop gain of individual loop-1, $P_{11} = -G_3G_4H_1$

Loop gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Loop gain of individual loop-3, $P_{31} = -G_1G_2G_3G_4$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$
$$= 1 - (-G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4)$$
$$= 1 + G_3G_4H_1 + G_2G_3H_2 + G_1G_2G_3G_4$$

Since no part of the graph is non-touching with forward path-1, $\Delta_1 = 1$.

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}P_1\Delta_1 \text{ (Number of forward path is 1 and so K = 1)}$$

$$= \frac{G_1G_2G_3G_4}{1+G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4}$$

EXAMPLE 1.29

The signal flow graph for a feedback control system is shown in fig 1. Determine the closed loop transfer function C(s)/R(s).

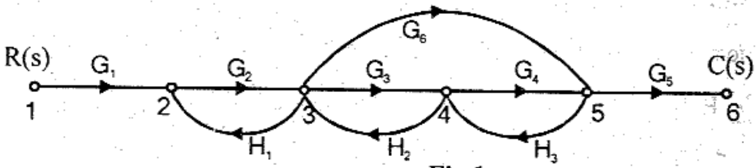


Fig 1

SOLUTION

1. Forward Path Gains

There are two forward paths. $\therefore$ K = 2.

Let forward path gains be P_1 and P_2.

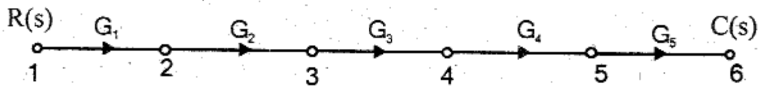


Fig 2 : Forward path-1

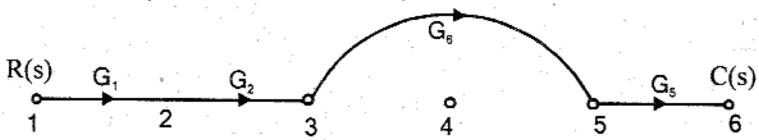


Fig 3 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2G_3G_4G_5$

Gain of forward path-2, $P_2 = G_1G_2G_6G_5$

2. Individual Loop Gain

There are four individual loops. Let individual loop gains be P_{11}, P_{21}, P_{31} and P_{41}.

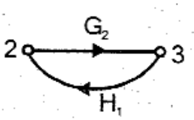


Fig 4 : loop-1

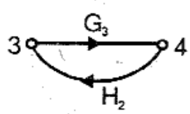


Fig 5 : loop-2

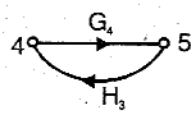


Fig 6 : loop-3

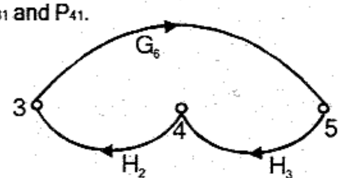


Fig 7 : loop-4

Loop gain of individual loop-1, $P_{11} = G_2H_1$

Loop gain of individual loop-2, $P_{21} = G_3H_2$

Loop gain of individual loop-3, $P_{31} = G_4H_3$

Loop gain of individual loop-4, $P_{41} = G_6H_2H_3$

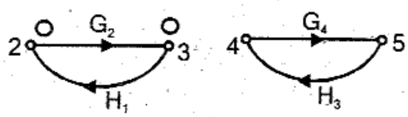


Fig 8 : First combination of two non touching loops

3. Gain Products of Two Non-touching Loops

There is only one combination of two non-touching loops. Let the gain products of two non-touching loops be P_{12}.

$$\left.\begin{array}{l}\text{Gain product of first combination}\\ \text{of two non - touching loops}\end{array}\right\} P_{12} = (G_2H_1)\,(G_4H_3)$$
$$= G_2G_4H_1H_3$$

IV. Calculation of Δ and Δ_K

$$\begin{aligned}\Delta &= 1-(P_{11}+P_{21}+P_{31}+P_{41})+P_{12}\\ &= 1-(G_2H_1+G_3H_2+G_4H_3+G_6H_2H_3)+G_2G_4H_1H_3\\ &= 1-G_2H_1-G_3H_2-G_4H_3-G_6H_2H_3+G_2G_4H_1H_3\end{aligned}$$

Since there is no part of graph which is non-touching with forward path-1 and 2, $\Delta_1=\Delta_2=1$

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T=\frac{1}{\Delta}\sum_K P_K\Delta_K=\frac{1}{\Delta}(P_1\Delta_1+P_2\Delta_2) \quad \text{(Number of forward paths is two and so K = 2)}$$
$$=\frac{G_1G_2G_3G_4G_5+G_1G_2G_5G_6}{1-G_2H_1-G_3H_2-G_4H_3-G_6H_2H_3+G_2G_4H_1H_3}$$

EXAMPLE 1.30

Convert the given block diagram to signal flow graph and determine C(s)/R(s).

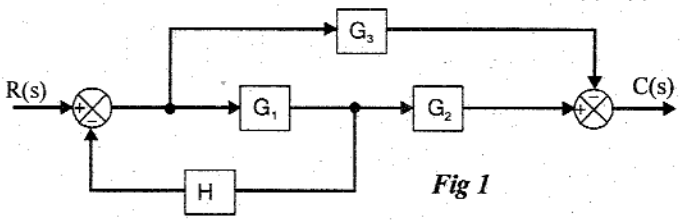


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

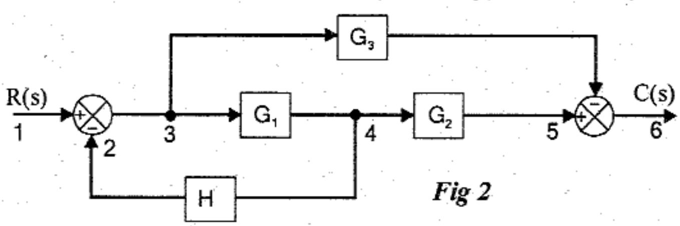


Fig 2

The signal flow graph of the above system is shown in fig 3.

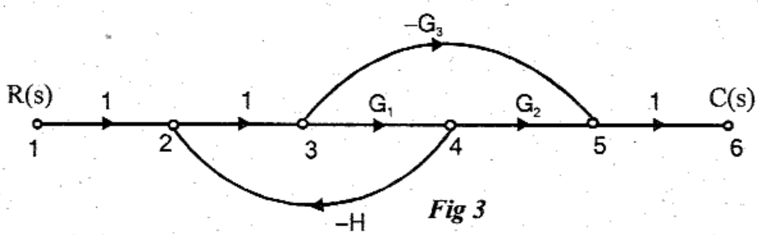


Fig 3

I. Forward Path Gains

There are two forward paths. ∴ K=2

Let the forward path gains be P_1 and P_2.

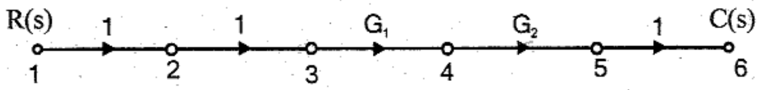


Fig 4 : Forward path-1

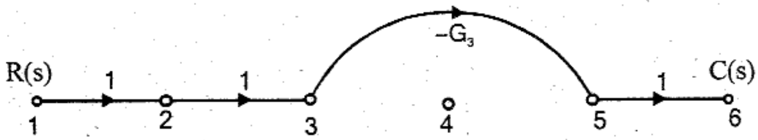


Fig 5 : Forward path-2

Gain of forward path-1, $P_1 = G_1G_2$

Gain of forward path-2, $P_2 = -G_3$

Individual Loop Gain

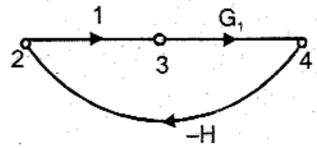


Fig 3 : loop-1

There is only one individual loop. Let the individual loop gain be P_{11}.

Loop gain of individual loop-I, $P_{11} = -G_1H$.

Gain Products of Two Non-touching Loops

There are no combinations of non-touching Loops.

Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11}] = 1 + G_1H$$

Since there are no part of the graph which is non-touching with forward path-1 and 2,

$$\Delta_1 = \Delta_2 = 1$$

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] = \frac{G_1G_2 - G_3}{1 + G_1H}$$

EXAMPLE 1.31

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

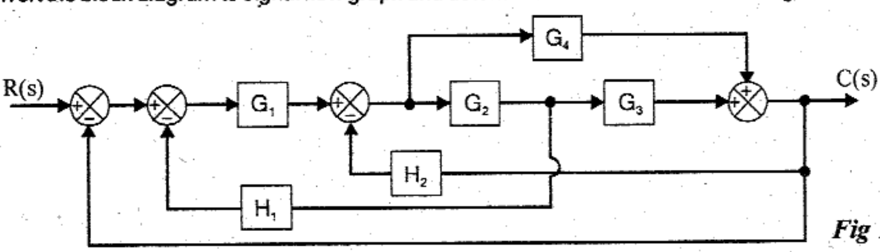


Fig 1

SOLUTION

The nodes are assigned at input, ouput, at every summing point & branch point as shown in fig 2.

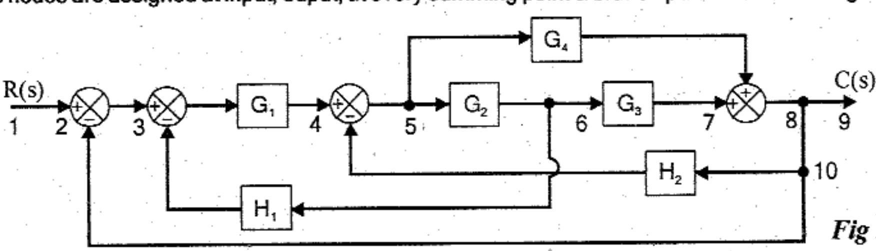


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

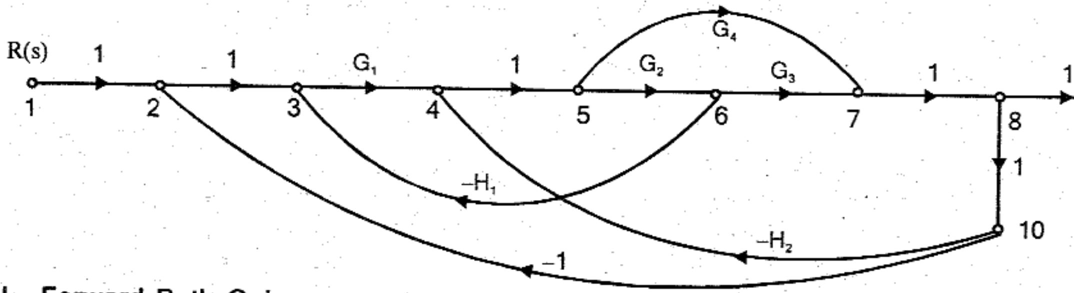


Fig 3

I. Forward Path Gains

There are two forward paths. ∴ K=2.

Let the gain of the forward paths be P_1 and P_2.

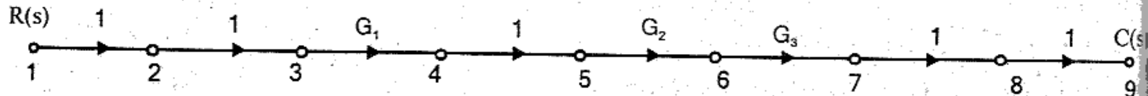
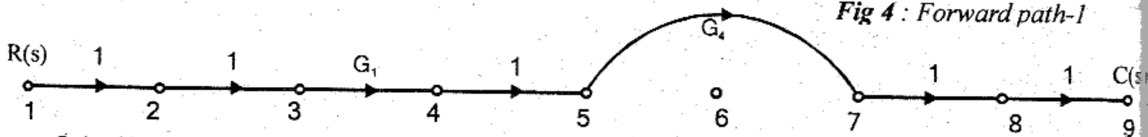


Fig 4 : Forward path-1



Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_1G_4$

Fig 5 : Forward path-2

II. Individual Loop Gain

There are five individual loops. Let the individual loop gain be P_{11}, P_{21}, P_{31}, P_{41} and P_{51}.

Loop gain of individual loop-1, $P_{11} = -G_1G_2G_3$

Loop gain of individual loop-2, $P_{21} = -G_2G_1H_1$

Loop gain of individual loop-3, $P_{31} = -G_2G_3H_2$

Loop gain of individual loop-4, $P_{41} = -G_1G_4$

Loop gain of individual loop-5, $P_{51} = -G_4H_2$

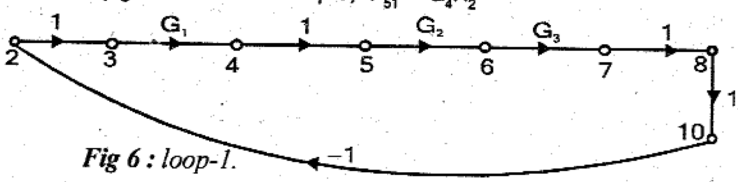


Fig 6 : loop-1.

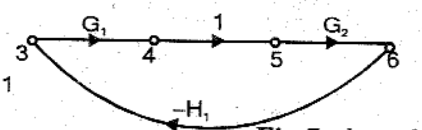


Fig 7 : loop-2.

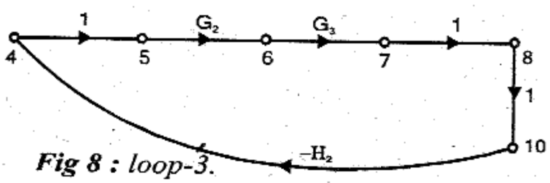


Fig 8 : loop-3.

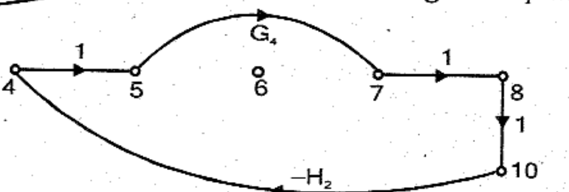


Fig 10 : loop-5.

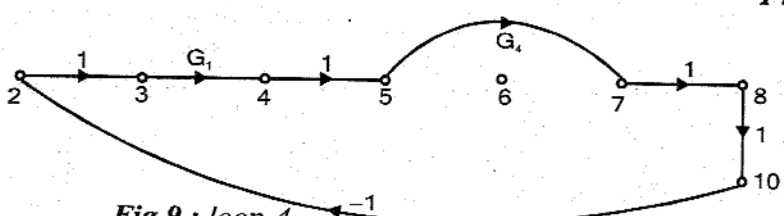


Fig 9 : loop-4.

Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

Calculation of Δ and Δ_K

$$\Delta = 1-[P_{11}+P_{21}+P_{31}+P_{41}+P_{51}] = 1+G_1G_2G_3+G_1G_2H_1+G_2G_3H_2+G_1G_4+G_4H_2$$

Since no part of graph is non touching with forward paths-I and 2, $\Delta_1 = \Delta_2 = 1$.

Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2]$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2}$$

EXAMPLE 1.32

Convert the block diagram to signal flow graph and determine the transfer function using Mason's gain formula.

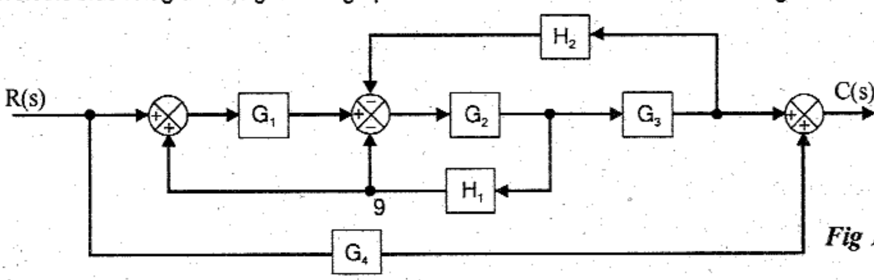


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

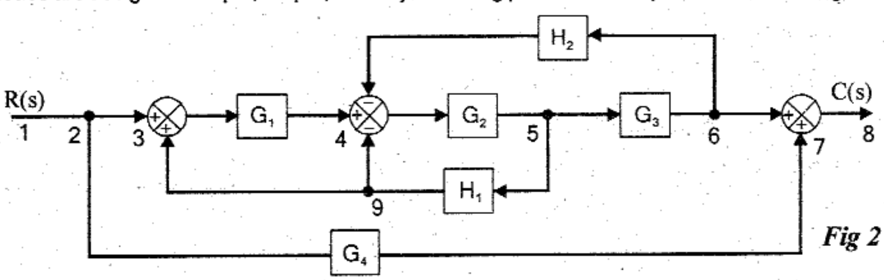


Fig 2

The signal flow graph for the above block diagram is shown in fig 3.

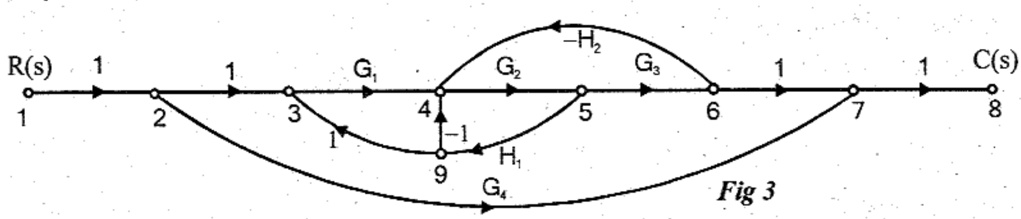


Fig 3

Forward Path Gains

There are two forward path, $\therefore$ K=2.

Let the forward path gains be P_1 and P_2.

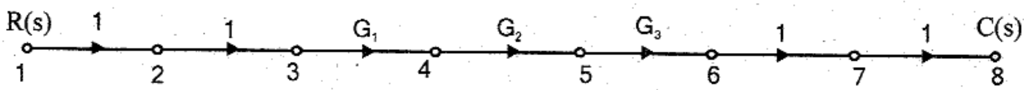


Fig 4 : Forward path-1.

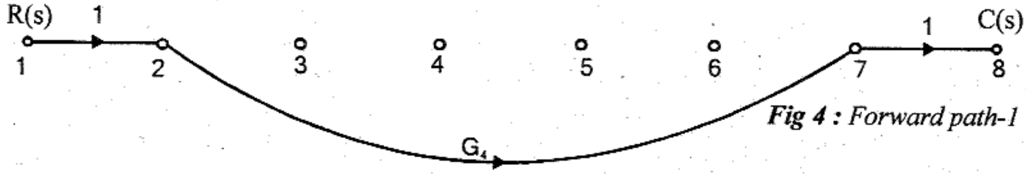


Fig 4 : Forward path-1

Gain of forward path-1, $P_1 = G_1G_2G_3$

Gain of forward path-2, $P_2 = G_4$

II. Individual Loop Gain

There are three individual loops with gains P_{11}, P_{21} and P_{31}.

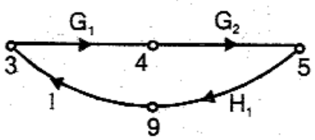


Fig 6 : loop-1.

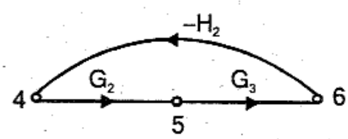


Fig 7 : loop-2.

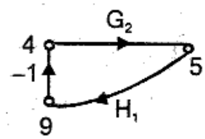


Fig 8 : loop-3.

Gain of individual loop-1, $P_{11} = G_1G_2H_1$

Gain of individual loop-2, $P_{21} = -G_2G_3H_2$

Gain of individual loop-3, $P_{31} = -G_2H_1$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two-non touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [P_{11} + P_{21} + P_{31}] = 1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

Since no part of graph touches forward path-1, $\Delta_1 = 1$.

The part of graph non touching forward path-2 is shown in fig 9.

$$\therefore \Delta_2 = 1 - [G_1G_2H_1 - G_2G_3H_2 - G_2H_1]$$
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1$$

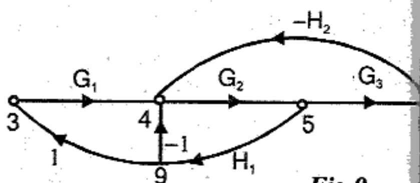


Fig 9

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_k P_k\Delta_k = \frac{1}{\Delta}[P_1\Delta_1 + P_2\Delta_2] \text{ (Number of forward paths is 2 and so K = 2)}$$

$$= \frac{1}{\Delta}\left[G_1G_2G_3 + G_4(1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1)\right]$$

$$= \frac{1}{\Delta}\left[G_1G_2G_3 + G_4 - G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1\right]$$

$$= \frac{G_1G_2G_3 + G_4 - G_1G_2G_4H_1 + G_2G_3G_4H_2 + G_2G_4H_1}{1 - G_1G_2H_1 + G_2G_3H_2 + G_2H_1}$$

EXAMPLE 1.33

Draw a signal flow graph and evaluate the closed loop transfer function of a system whose block diagram is shown in fig 1.

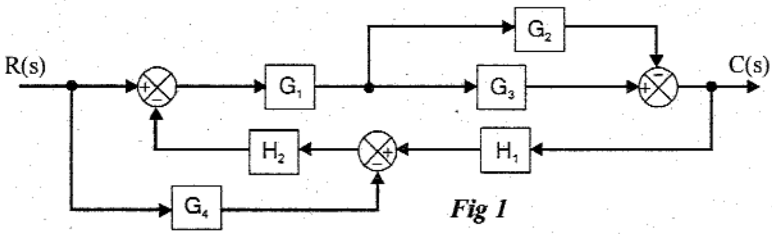


Fig 1

SOLUTION

The nodes are assigned at input, output, at every summing point & branch point as shown in fig 2.

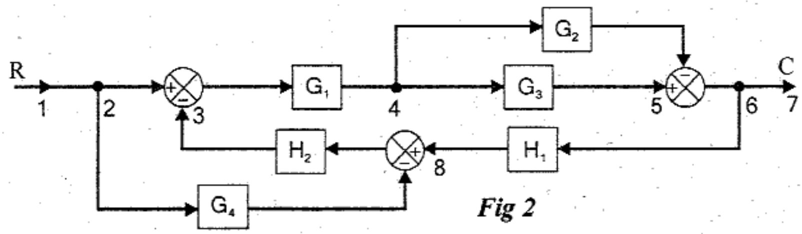


Fig 2

The signal flow graph for the block diagram of fig 2, is shown in fig 3.

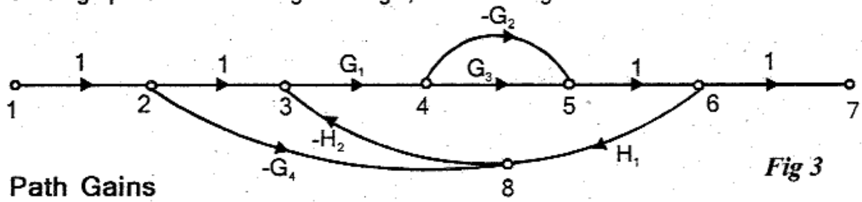


Fig 3

Forward Path Gains

There are four forward paths, $\therefore K = 4$

Let the forward path gains be P_1, P_2, P_3 and P_4.

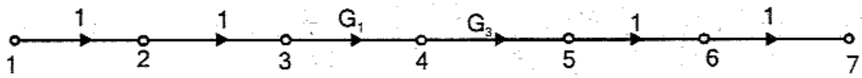


Fig 4 : Forward path-1.

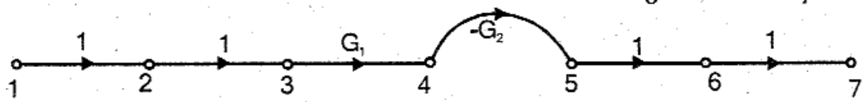


Fig 5 : Forward path-2.

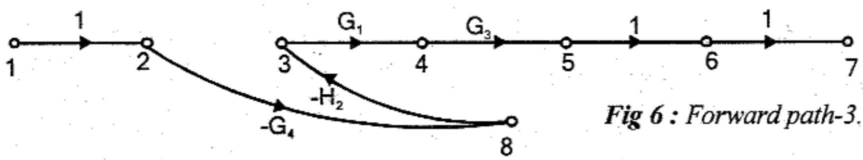


Fig 6 : Forward path-3.

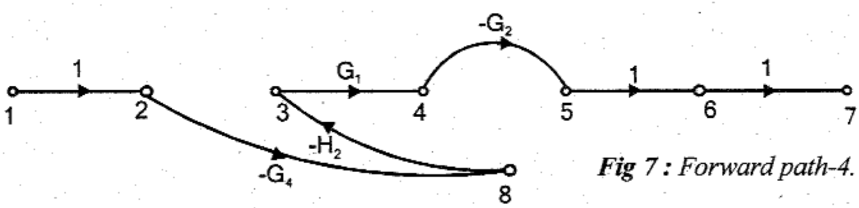


Fig 7 : Forward path-4.

Gain of forward path-1, $P_1 = G_1G_3$
Gain of forward path-2, $P_2 = -G_1G_2$
Gain of forward path-3, $P_3 = G_1G_3G_4H_2$
Gain of forward path-4, $P_4 = -G_1G_2G_4H_2$

II. Individual Loop Gain

There are two individual loops, let individual loop gains be P_{11} and P_{21}.

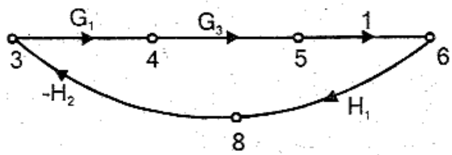


Fig 7 : loop-1

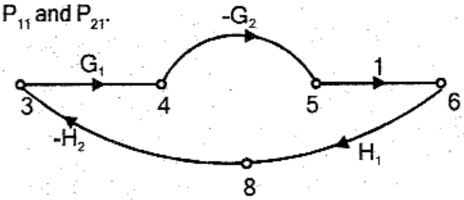


Fig 7 : loop-2

Loop gain of individual loop-1, $P_{11} = -G_1G_3H_1H_2$
Loop gain of individual loop-2, $P_{21} = G_1G_2H_1H_2$

III. Gain Products of Two Non-touching Loops

There are no possible combinations of two non-touching loops, three non-touching loops, etc.,.

IV. Calculation of Δ and Δ_K

$$\Delta = 1 - [\text{sum of individual loop gain}] = 1 - (P_{11} + P_{21})$$
$$= 1 - [-G_1G_3H_1H_2 + G_1G_2H_1H_2] = 1 + G_1G_3H_1H_2 - G_1G_2H_1H_2$$

Since no part of graph is non touching with the forward paths, $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$.

V. Transfer Function, T

By Mason's gain formula the transfer function, T is given by,

$$T = \frac{1}{\Delta}\sum_K P_K\Delta_K = \frac{P_1 + P_2 + P_3 + P_4}{\Delta} \text{ (Number of forward paths is 4 and so K = 4)}$$

$$= \frac{G_1G_3 - G_1G_2 + G_1G_3G_4H_2 - G_1G_2G_4H_2}{1 + G_1G_3H_1H_2 - G_1G_2H_1H_2}$$

$$= \frac{G_1(G_3 - G_2) + G_1G_4H_2(G_3 - G_2)}{1 + G_1H_1H_2(G_3 - G_2)} = \frac{G_1(G_3 - G_2)(1 + G_4H_2)}{1 + G_1H_1H_2(G_3 - G_2)}$$

1.14 SHORT QUESTIONS AND ANSWERS

Q1.1 What is system?

When a number of elements or components are connected in a sequence to perform a specific function, the group thus formed is called a system.

Q1.2 What is control system?

A system consists of a number of components connected together to perform a specific function. In a system when the output quantity is controlled by varying the input quantity, then the system is called control system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Q1.3 What are the two major type of control systems?

The two major type of control systems are open loop and closed loop systems.

Q1.4 Define open loop system.

The control system in which the output quantity has no effect upon the input quantity are called open loop control system. This means that the output is not fedback to the input for correction.

Q1.5 *Define closed loop system.*

The control systems in which the output has an effect upon the input quantity in order to maintain the desired output value are called closed loop control systems.

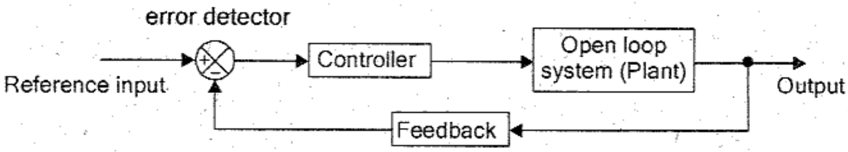
Q1.6 *What is feedback? What type of feedback is employed in control system?*

The feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in desired output.

Negative feedback is employed in control system.

Q1.7 *What are the components of feedback control system?*

The components of feedback control system are plant, feedback path elements, error detector and controller.



Q1.8 *Why negative feedback is invariably preferred in a closed loop system?*

The negative feedback results in better stability in steady state and rejects any disturbance signals. It also has low sensitivity to parameter variations. Hence negative feedback is preferred in closed loop systems.

Q1.9 *What are the characteristics of negative feedback?*

The characteristics of negative feedback are as follows :

(i) accuracy in tracking steady state value.

(ii) rejection of disturbance signals.

(iii) low sensitivity to parameter variations.

(iv) reduction in gain at the expense of better stability.

Q1.10. *What is the effect of positive feedback on stability?*

The positive feedback increases the error signal and drives the output to instability. But sometimes the positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

Q1.11. *Distinguish between open loop and closed loop system.*

Open loop	Closed loop
1. Inaccurate & unreliable.	1. Accurate & reliable.
2. Simple and economical.	2. Complex and costly.
3. Changes in output due to external disturbances are not corrected automatically.	3. Changes in output due to external disturbances are corrected automatically.
4. They are generally stable.	4. Great efforts are needed to design a stable system.

Q1.12 *What is servomechanism?*

The servomechanism is a feedback control system in which the output is mechanical position (or time derivatives of position e.g. velocity and acceleration).

Q1.13 *State the principle of homogenity (or) State the principle of superposition.*

The principle of superposition and homogenity states that if the system has responses $c_1(t)$ and $c_2(t)$ for the inputs $r_1(t)$ and $r_2(t)$ respectively then the system response to the linear combination of these input $a_1r_1(t) + a_2r_2(t)$ is given by linear combination of the individual outputs $a_1c_1(t) + a_2c_2(t)$, where a_1 and a_2 are constants.

Q1.14 ***Define linear system.***

A system is said to be linear, if it obeys the principle of superposition and homogenity, which states that the response of a system to a weighed sum of signals is equal to the corresponding weighed sum of the responses of the system to each of the individual input signals. The concept of linear system is diagrammatically shown below.

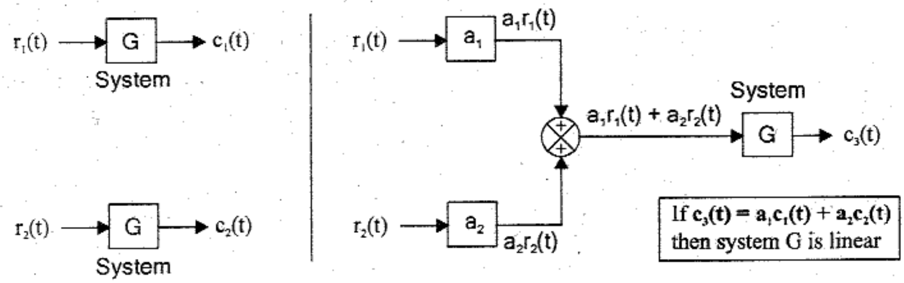


Fig Q1.14 : Principle of linearity and superposition.

Q1.15 ***What is time invariant system?***

A system is said to be time invariant if its input-output characteristics do not change with time. A linear time invariant system can be represented by constant coefficient differential equations. (In linear time varying systems the coefficients of the differential equation governing the system are function of time).

Q1.16 ***Define transfer function.***

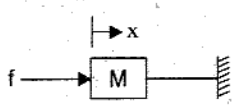
The transfer function of a system is defined as the ratio of Laplace transform of output to Laplace transform of input with zero initial conditions. (It is also defined as the Laplace transform of the impulse response of system with zero initial conditions).

Q1.17 ***What are the basic elements used for modelling mechanical translational system?***

The model of mechanical translational system can be obtained by using three basic elements mass, spring and dashpot.

Q1.18 ***Write the force balance equation of ideal mass element.***

Let a force f be applied to an ideal mass M. The mass will offer an opposing force, f_m which is proportional to acceleration.

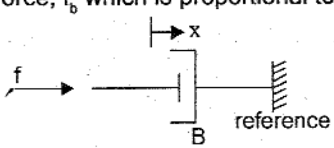


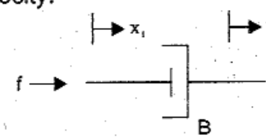
$$\therefore\ f = f_m = M\frac{d^2x}{dt^2}$$

Q1.19 ***Write the force balance equation of ideal dashpot.***

Let a force f be applied to an ideal dashpot, with viscous frictional coefficient B. The dashpot will offer an opposing force, f_b which is proportional to velocity.

$$f = f_b = B\frac{dx}{dt}$$

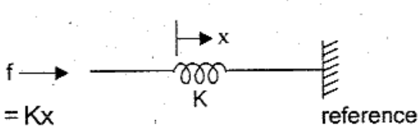




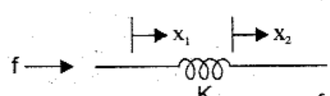
$$f = f_b = B\frac{d}{dt}(x_1 - x_2)$$

Q1.20 ***Write the force balance equation of ideal spring.***

Let a force f be applied to an ideal spring with spring constant K. The spring will offer an opposing force, f_k which is proportional to displacement.



$$f = f_k = Kx$$



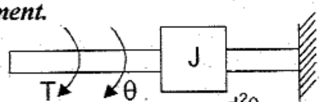
$$f = f_k = K(x_1 - x_2)$$

1.21 ***What are the basic elements used for modelling mechanical rotational system?***

The model of mechanical rotational system can be obtained using three basic elements mass with moment of inertia, J, dash-pot with rotational frictional coefficient, B and torsional spring with stiffness, K.

1.22 ***Write the torque balance equation of an ideal rotational mass element.***

Let a torque T be applied to an ideal mass with moment of inertia, J. The mass will offer an opposing torque T_j which is proportional to angular acceleration.

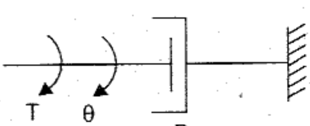


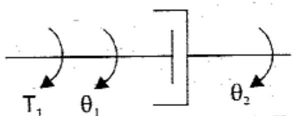
$$T = T_j = J\frac{d^2\theta}{dt^2}$$

1.23 ***Write the torque balance equation of an ideal rotational dash-pot.***

Let a torque T be applied to a rotational dash-pot with frictional coefficient B. The dashpot will offer an opposing torque which is proportional to angular velocity.

$$T = T_b = B\frac{d\theta}{dt}$$



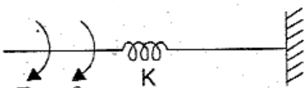


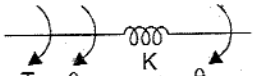
$$T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

1.24 ***Write the torque balance equation of ideal rotational spring.***

Let a torque T be applied to an ideal rotational spring with spring constant K. The spring will offer an opposing torque T_k which is proportional to angular displacement.

$$T_k = K\theta$$





$$T = T_k = K(\theta_1 - \theta_2)$$

1.25 ***Name the two types of electrical analogous for mechanical system.***

The two types of analogies for the mechanical system are force-voltage and force-current analogy.

1.26 ***Write the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system.***

Force, f	→	Voltage, e	Frictional coefficient , B	→	Resistance, R
Velocity, v	→	Current, i	Stiffness, K	→	Inverse of capacitance, 1/C
Displacement, x	→	Charge, q	Newton's second law, $\Sigma f = 0$	→	Kirchoff 's voltage law, $\Sigma v = 0$
Mass, M	→	Inductance, L			

1.27 ***Write the analogous electrical elements in force-current analogy for the elements of mechanical translational system.***

Force, f	→	Current, i	Frictional coefficient, B	→	Conductance, G =1/ R
Velocity, v	→	Voltage, v	Stiffness, K	→	Inverse of Inductance, 1/L
Displacement, x	→	Flux, ϕ	Newton's second law, $\Sigma f = 0$	→	Kirchoff 's current law, $\Sigma i = 0$
Mass, M	→	Capacitance, C			

1.28 ***Write the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system.***

Torque, T	→	Voltage, e	Stiffness of spring, K	→	Inverse of capacitance, 1/C
Angular velocity, ω	→	Current, i	Frictional coefficient, B	→	Resistance, R
Moment of inertia, J	→	Inductance, L	Newton's second law, $\Sigma T = 0$	→	Kirchoff 's voltage law, $\Sigma v = 0$.
Angular displacement, θ	→	Charge, q			

Q1.29 ***Write the analogous electrical elements in torque-current analogy for the elements of mechanical rotational system.***

Torque, T → Current, i	Frictional coefficient, B → Conductance, G = 1/R
Angular velocity, ω → Voltage, v	Stiffness of spring, K → Inverse of inductance, 1/L
Angular displacement, θ → Flux, ϕ	Newton's second law, $\Sigma T = 0$ → Kirchoff 's current law, $\Sigma i = 0$
Moment of inertia, J → Capacitance, C	

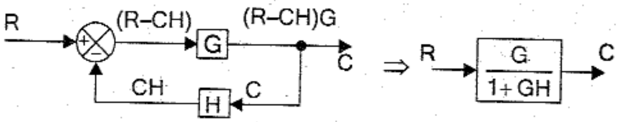
Q1.30 ***What is block diagram? What are the basic components of block diagram?***

A block diagram of a system is a pictorial representation of the functions performed by each component of the system and shows the flow of signals. The basic elements of block diagram are block, branch point and summing point.

Q1.31 ***What is the basis for framing the rules of block diagram reduction technique?***

The rules for block diagram reduction technique are framed such that any modification made on the diagram does not alter the input output relation.

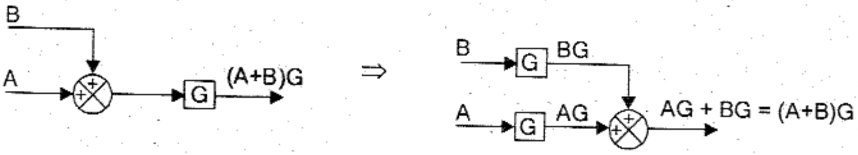
Q1.32 ***Write the rule for eliminating negative feedback loop.***



Proof

$$C = (R - CH)\,G$$
$$C = RG - CHG$$
$$C + CHG = RG$$
$$C(1 + HG) = RG$$
$$\frac{C}{R} = \frac{G}{1+GH}$$

Q1.33 ***Write the rule for moving the summing point ahead of a block.***



Q1.34 ***What is a signal flow graph?***

A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. By taking Laplace transform, the time domain differential equations governing a control system can be transferred to a set of algebraic equations in s-domain. The signal flow graph of the system can be constructed using these equations.

Q1.35 ***What is transmittance?***

The transmittance is the gain acquired by the signal when it travels from one node to another node in signal flow graph.

Q1.36 ***What is sink and source?***

Source is the input node in the signal flow graph and it has only outgoing branches. Sink is a output node in the signal flow graph and it has only incoming branches.

Q1.37 ***Define non-touching loop.***

The loops are said to be non-touching if they do not have common nodes.

Q1.38 ***What are the basic properties of signal flow graph?***

The basic properties of signal flow graph are,

(i) Signal flow graph is applicable to linear systems.

(ii) It consists of nodes and branches. A node is a point representing a variable or signal. A branch indicates functional dependence of one signal on the other.

(iii) A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

(iv) Signals travel along branches only in the marked direction and when it travels it gets multiplied by the gain or transmittance of the branch.

(v) The algebraic equations must be in the form of cause and effect relationship.

Q1.39 ***Write the Mason's gain formula.***

Mason's gain formula states that the overall gain of the system [transfer function] as follows,

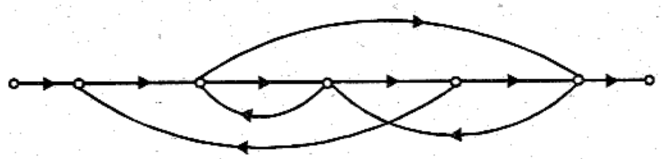
Overall gain, $T = \frac{1}{\Delta} \sum_K P_K \Delta_K$

$T = T(s)$ = Transfer function of the system
K = Number of forward paths in the signal flow graph
P_K = Forward path gain of K^{th} forward path

$$\Delta = 1 - \begin{bmatrix}\text{sum of individual}\\ \text{loop gains}\end{bmatrix} + \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of two non-touching loops}\end{bmatrix}$$
$$- \begin{bmatrix}\text{sum of gain products of all possible}\\ \text{combinations of three non-touching loops}\end{bmatrix} + \ldots\ldots$$

$\Delta_K = \Delta$ for that part of the graph which is not touching K^{th} forward path

Q1.40 ***For the given signal flow graph, identify the number of forward path and number of individual loop.***



Number of forward paths = 2
Number of individual loops = 4

1.15 EXERCISES

E1.1 For the mechanical system shown in fig E1.1derive the transfer function. Also draw the force-voltage and force-current analogous circuits.

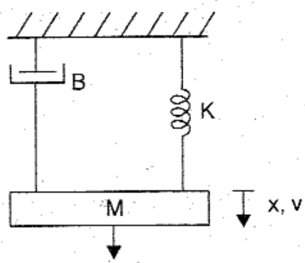


Fig E1.1

E1.2 For the mechanical system shown in fig E1.2 draw the force-voltage and force-current analogous circuits.

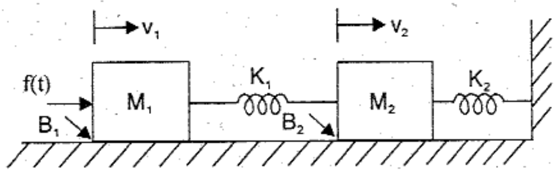


Fig E1.2

E1.3 Write the differential equations governing the mechanical system shown in fig E1.3(a) & (b). Also draw the force-voltage and force-current analogous circuit.

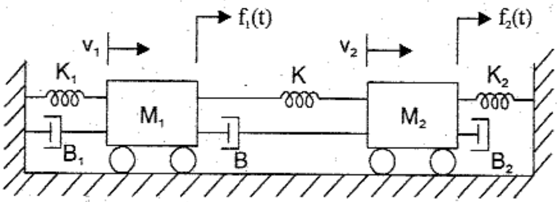


Fig E1.3(a)

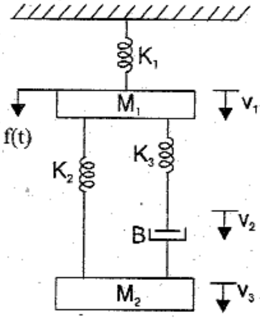


Fig E1.3(b)

E1.4 Consider the mechanical translational system shown in fig E1.4, Draw(a) force-voltage and (b) force-current analogous circuits.

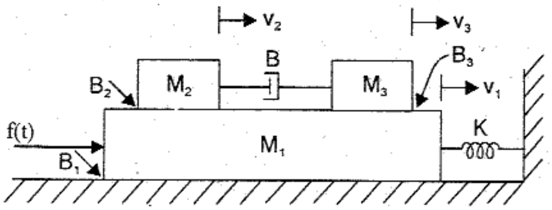


Fig E1.4

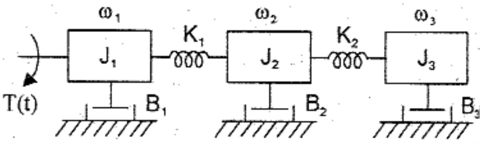


Fig E1.5

E1.5 Write the differential equations governing the rotational mechanical system shown in fig E1.5. Also draw the torque-voltage and torque-current analogous circuits.

E1.6 In an electrical circuit the elements resistance, capacitance and inductance are connected in parallel across the voltage source E as shown in fig E1.6, Draw(a) Translation mechnical analogous system (b) Rotational mechanical analogous system.

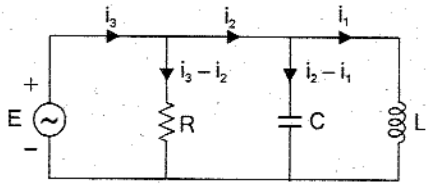


Fig E1.6

E1.7 Consider the block diagram shown in fig E.1.7(a), (b) (c) & (d). Using the block diagram reduction technique, find C/R.

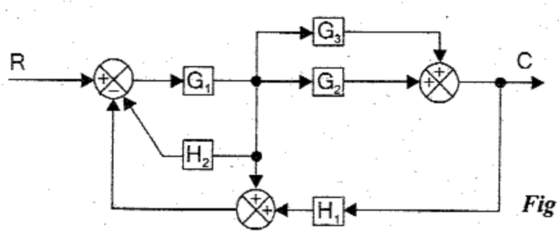


Fig E1.7(a)

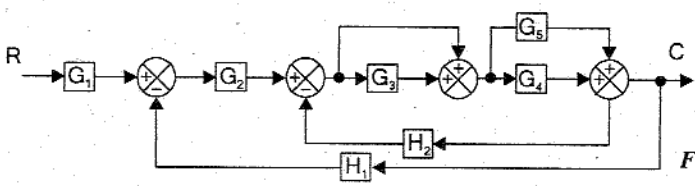


Fig E1.7(b)

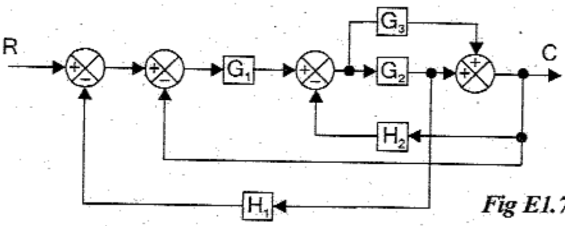


Fig E1.7(c)

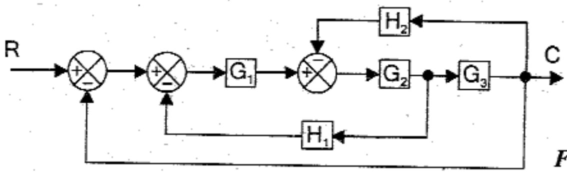


Fig E1.7(d)

E 8 Convert the block diagram shown in fig E1.8 to signal flow graph and find the transfer function of the system.

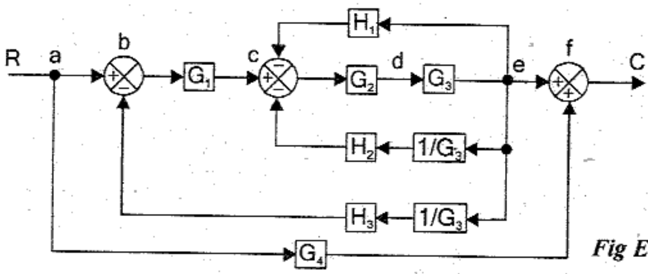


Fig E1.8

E 9 Consider the system shown in fig E1.9(a), (b), (c) & (d). obtain the transfer function using Mason's gain formula.

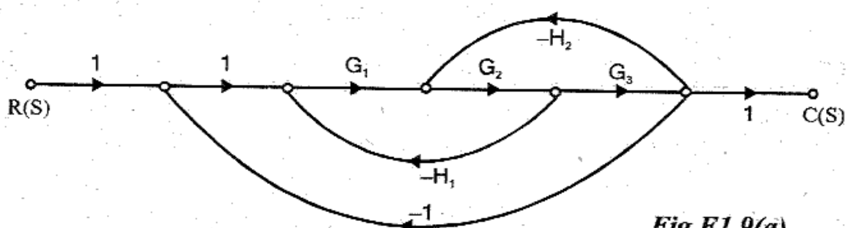


Fig E1.9(a)

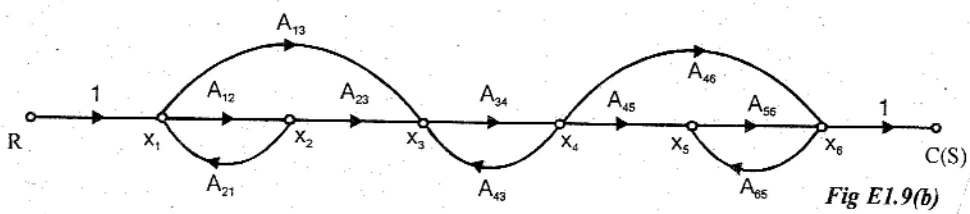


Fig E1.9(b)

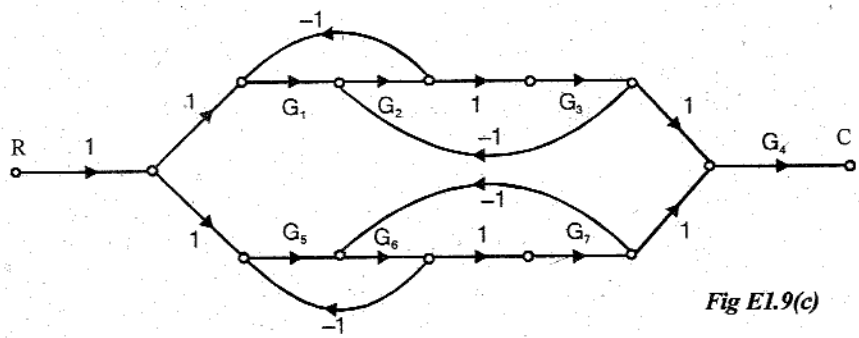


Fig E1.9(c)

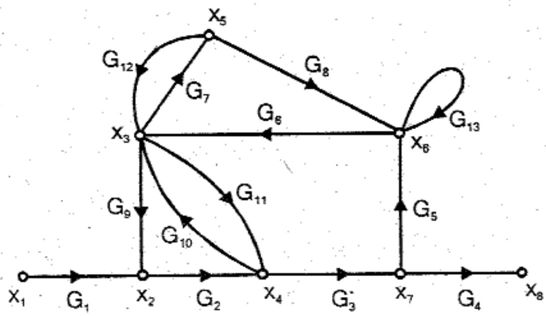


Fig E1.9(d)

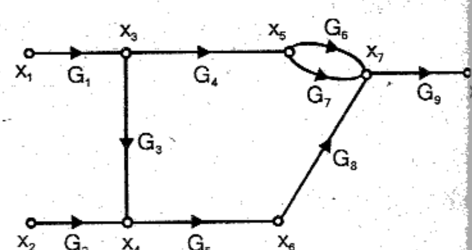


Fig E1.10

E1.10 Consider the signal flow graph shown in fig E.1.10 obtain $\frac{X_8}{X_1}$ and $\frac{X_8}{X_2}$

E1.11 Find the transfer functions of the networks shown in fig E1.11(a), (b), (c) & (d).

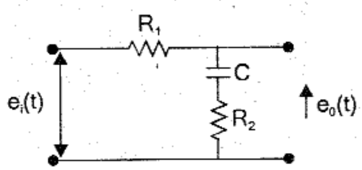


Fig E1.11(a)

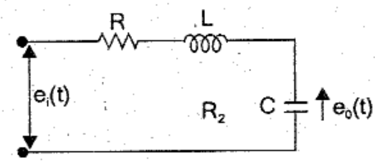


Fig E1.11(b)

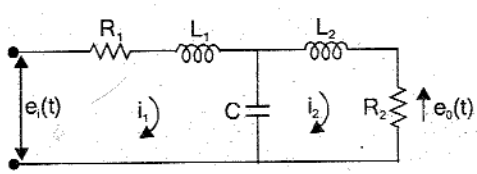


Fig E1.11(c)

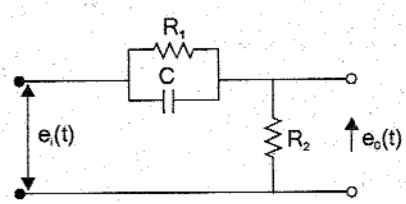


Fig E1.11(d)

E1.12 Find the transfer function of the circuit shown in fig E1.12.

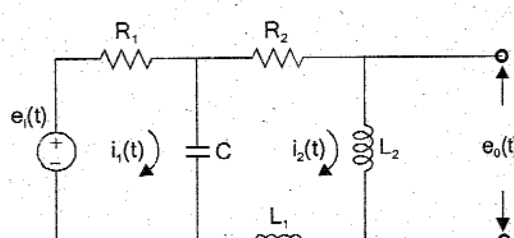
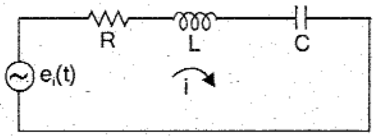


Fig E1.12

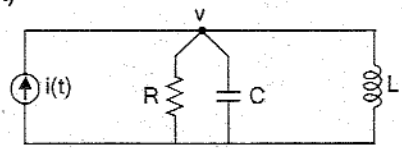
ANSWER FOR EXERCISE PROBLEMS

1\. The transfer function is $\dfrac{X(s)}{F(s)} = \dfrac{1}{(Ms^2 + Bs + K)}$



$f(t) \to e(t)$ $\quad M \to L$ $\quad K \to 1/C$
$v \to i$ $\quad B \to R$

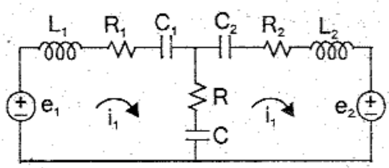
Force-voltage analogous circuit



$f(t) \to i(t)$ $\quad M \to C$ $\quad K \to 1/L$
$v \to v$ $\quad B \to 1/R$

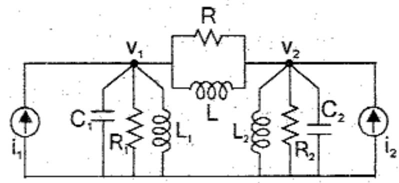
Force-current analogous circuit

2\.



$f_1 \to e_1$ $\quad M_1 \to L_1$ $\quad B \to R$
$f_2 \to e_2$ $\quad M_2 \to L_2$ $\quad K_1 \to 1/C_1$
$v_1 \to i_1$ $\quad B_1 \to R_1$ $\quad K_2 \to 1/C_2$
$v_2 \to i_2$ $\quad B_2 \to R_2$ $\quad K \to 1/C$

Force-voltage analogous circuit

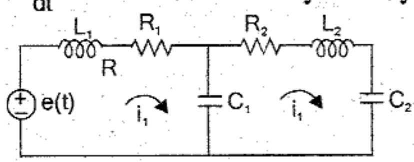


$f_1 \to i_1$ $\quad M_1 \to C_1$ $\quad B \to 1/R$
$f_2 \to i_2$ $\quad M_2 \to C_2$ $\quad K_1 \to 1/L_1$
$v_1 \to v_1$ $\quad B_1 \to 1/R_1$ $\quad K_2 \to 1/L_2$
$v_2 \to v_2$ $\quad B_2 \to 1/R_2$ $\quad K \to 1/L$

Force-current analogous circuit

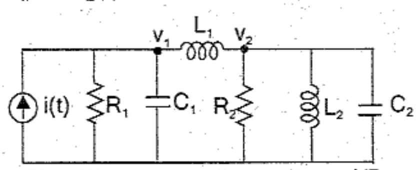
3\. a) $M_1 \dfrac{dv_1}{dt} + B_1 v_1 + B(v_1 - v_2) + K_1 \int v_1 dt + K \int (v_1 - v_2) dt = f_1(t)$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + B(v_2 - v_1) + K_2 \int v_2 dt + K \int (v_2 - v_1) dt = f_2(t)$$



$f(t) \to e(t)$ $\quad M_1 \to L_1$ $\quad B_2 \to R_2$
$v_1 \to i_1$ $\quad M_2 \to L_2$ $\quad K_1 \to 1/C_1$
$v_2 \to i_2$ $\quad B_1 \to R_1$ $\quad K_2 \to 1/C_2$

Force-voltage analogous circuit

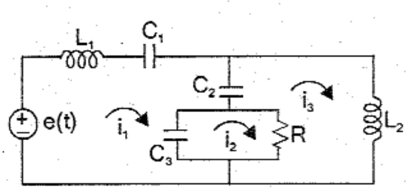


$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad B_2 \to 1/R_2$
$v_1 \to v_1$ $\quad M_2 \to C_2$ $\quad K_1 \to 1/L_1$
$v_2 \to v_2$ $\quad B_1 \to 1/R_1$ $\quad K_2 \to 1/L_2$

Force-current analogous circuit

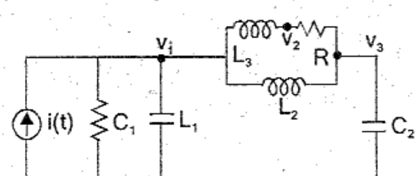
3\. b) $M_1 \dfrac{dv_1}{dt} + K_1 \int v_1 dt + K_2 \int (v_1 - v_3) dt + K_3 \int (v_1 - v_2) dt = f(t)$

$K_3 \int (v_2 - v_1) dt + B(v_2 - v_3) = 0;$ $\qquad M_2 \dfrac{dv_3}{dt} + B(v_3 - v_2) + K_2 \int (v_3 - v_1) dt = 0$



$f(t) \to e(t)$ $\quad M_1 \to L_1$ $\quad K_1 \to 1/C_1$
$v_1 \to i_1$ $\quad M_2 \to L_2$ $\quad K_2 \to 1/C_2$
$v_2 \to i_2$ $\quad B \to R$ $\quad K_3 \to 1/C_3$
$v_3 \to i_3$

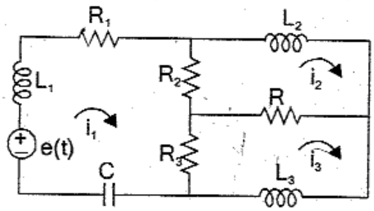
Force-voltage analogous circuit



$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad K_1 \to 1/L_1$
$v_1 \to v_1$ $\quad M_2 \to C_2$ $\quad K_2 \to 1/L_2$
$v_2 \to v_2$ $\quad B \to 1/R$ $\quad K_3 \to 1/L_3$
$v_3 \to v_3$

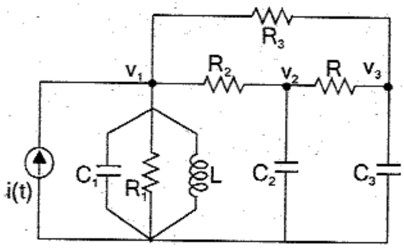
Force-current analogous circuit

E1.4



$f(t) \to e(t)$ $M_1 \to L_1$ $B_1 \to R_1$
$v_1 \to i_1$ $M_2 \to L_2$ $B_2 \to R_2$
$v_2 \to i_2$ $M_3 \to L_3$ $B_3 \to R_3$
$v_3 \to i_3$ $B \to R$ $K \to 1/C$

Force voltage-analogous circuit

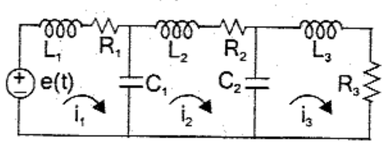


$f(t) \to i(t)$ $M_1 \to C_1$ $B_1 \to 1/R_1$
$v_1 \to v_1$ $M_2 \to C_2$ $B_2 \to 1/R_2$
$v_2 \to v_2$ $M_3 \to C_3$ $B_3 \to 1/R_3$
$v_3 \to v_3$ $B \to 1/R$ $K \to 1/L$

Force-current analogous circuit

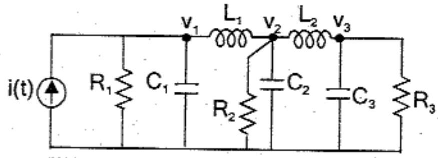
E1.5 $J_1 \dfrac{d\omega_1}{dt} + B_1\omega_1 + K_1\int(\omega_1 - \omega_2)dt = T(t)$; $\quad J_2 \dfrac{d\omega_2}{dt} + B_2\omega_2 + K_1\int(\omega_2 - \omega_1)dt + K_2\int(\omega_2 - \omega_3)dt = 0$

$$J_3 \frac{d\omega_3}{dt} + B_3\omega_3 + K_2\int(\omega_3 - \omega_2)dt = 0$$



$T(t) \to e(t)$ $J_1 \to L_1$ $B_1 \to R_1$ $K_1 \to 1/C_1$
$\omega_1 \to i_1$ $J_2 \to L_2$ $B_2 \to R_2$ $K_2 \to 1/C_2$
$\omega_2 \to i_2$ $J_3 \to L_3$ $B_3 \to R_3$ $\omega_3 \to i$

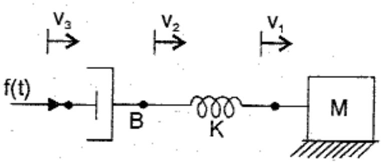
Torque-voltage analogous circuit



$T(t) \to i(t)$ $J_1 \to C_1$ $B_1 \to 1/R_1$ $K_1 \to 1/L_1$
$\omega_1 \to v_1$ $J_2 \to C_2$ $B_2 \to 1/R_2$ $K_2 \to 1/L_2$
$\omega_2 \to v_2$ $J_3 \to C_3$ $B_3 \to 1/R_3$ $\omega_3 \to v_3$

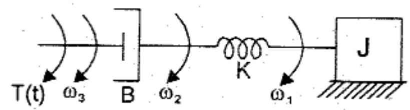
Torque-current analogous circuit

E1.6



$e(t) \to f(t)$ $i_1 \to v_1$ $i_3 \to v_3$ $R \to B$
$i_2 \to v_2$ $L \to M$ $1/C \to K$

Analogous mechanical translational system



$e(t) \to T(t)$ $i_1 \to \omega_1$ $i_3 \to \omega_3$ $R \to B$
$i_2 \to \omega_2$ $L \to J$ $1/C \to K$

Analogous mechanical rotational system

E1.7

(a) $$\frac{C}{R} = \frac{G_1G_2 + G_1G_3}{1 + G_1H_2 + G_1 + G_1G_2H_1 + G_1G_3H_1}$$

(b) $$\frac{C}{R} = \frac{G_1G_2(1+G_3)(G_4+G_5)}{1 + (1+G_3)(G_4+G_5)H_2 + (1+G_3)(G_4+G_5)G_2H_1}$$

(c) $$\frac{C}{R} = \frac{G_1G_2 + G_1G_3}{1 + G_1G_2H_1 + G_1G_2 + G_1G_3 + G_2H_2 + G_3H_2}$$

(d) $$\frac{C}{R} = \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_1G_2H_1 + G_1G_2G_3}$$

$$\frac{G_1G_2G_3 + G_4 + G_2G_3G_4H_1 + G_2G_4H_2 + G_1G_2G_4H_3}{1 + G_2G_3H_1 + G_2H_2 + G_1G_2H_3}$$

(a) $$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

(b) $$\frac{C}{R} = \frac{A_{12}A_{23}A_{34}A_{45}A_{56} + A_{13}A_{34}A_{45}A_{56} + A_{12}A_{23}A_{34}A_{46} + A_{13}A_{34}A_{46}}{1 - (A_{12}A_{21} + A_{34}A_{43} + A_{56}A_{65}) + (A_{12}A_{21}A_{34}A_{43} + A_{12}A_{21}A_{56}A_{65} + A_{34}A_{43}A_{56}A_{65}) - (A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})}$$

(c) $$\frac{C}{R} = \frac{G_1G_2G_3G_4\,(1 + G_5G_6 + G_6G_7) + G_4G_5G_6G_7(1 + G_1G_2 + G_2G_3)}{1 + G_1G_2 + G_2G_3 + G_5G_6 + G_6G_7 + G_1G_2G_5G_6 + G_5G_6G_2G_3 + G_1G_2G_6G_7 + G_2G_3G_6G_7}$$

(d) $$\frac{x_8}{x_1} = \frac{[G_1G_2G_3G_4]\,[1 - (G_7G_{12} + G_6G_7G_8 + G_{13}) + G_7G_{12}G_{13}]}{1 - [G_2G_9G_{10} + G_{10}G_{11} + G_2G_3G_5G_6G_9 + G_3G_5G_6G_{11} + G_7G_{12} + G_6G_7G_8 + G_{13}] + G_2G_9G_{10}G_{13} + G_{10}G_{11}G_{13} + G_7G_{12}G_{13}}$$

$$\frac{x_8}{x_1} = G_1G_4G_6G_9 + G_1G_4G_7G_9 + G_1G_3G_5G_8G_9 \quad ; \quad \frac{x_8}{x_2} = G_2G_5G_8G_9$$

(a) $$\frac{E_o(s)}{E_i(s)} = \frac{1 + s\,R_2\,C}{1 + s\,(R_1 + R_2)\,C}$$

(b) $$\frac{E_o(s)}{E_i(s)} = \frac{1}{s^2LC + s\,R\,C + 1}$$

(c) $$\frac{E_o(s)}{E_i(s)} = \frac{s\,R_2\,C}{(s^2L_1C + s\,R_1\,C + 1)\,(s^2L_2C + s\,R_2\,C + 1) - 1}$$

(d) $$\frac{E_o(s)}{E_i(s)} = \frac{sR_1R_2C + R_2}{sR_1R_2C + (R_1 + R_2)}$$

$$\frac{C(s)}{E(s)} = \frac{s^2\,L_2\,C}{[s\,R_1\,C + 1]\,[s^2(L_1 + L_2)\,C + s\,R_2\,C + 1] - 1}$$

CHAPTER 2

TIME RESPONSE ANALYSIS

2.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s)$ .....(2.1)

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

Response in s-domain, $C(s) = R(s)\,M(s)$ .....(2.2)

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\}$ .....(2.3)

where, $M(s) = \frac{G(s)}{1+G(s)H(s)}$

The time response of a control system consists of two parts : *the transient and the steady state response.* The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.

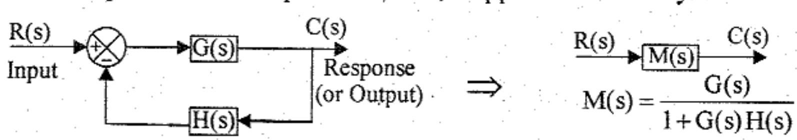


Fig 2.1 : Closed loop system.

2.2 TEST SIGNALS

The knowledge of input signal is required to predict the response of a system. In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

The standard test signals are,

1. a) Step signal
   b) Unit step signal
2. a) Ramp signal
   b) Unit ramp signal
3. a) Parabolic signal
   b) Unit parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Since the test signals are simple functions for time, they can be easily generated in laboratories. The mathematical and experimental analysis of control systems using these signals can be carried out easily. The use of the test signals can be justified because of a correlation existing between the response characteristics of a system to a test input signal and capability of the system to cope with actual input signals.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t = 0 and remains constant at A for t > 0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which **A** is unity.

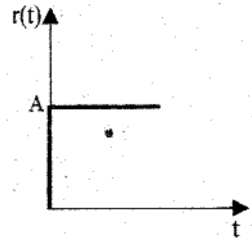


Fig 2.2 : Step signal.

The mathematical representation of the step signal is,

$$r(t) = 1 \; ; \; t \geq 0 \qquad\qquad .....(2.4)$$
$$= 0 \; ; \; t < 0$$

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of **A** is unity.

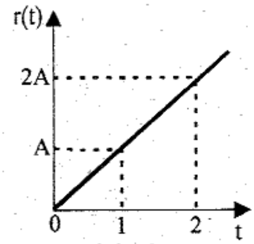


Fig 2.3 : Ramp signal.

The mathematical representation of the ramp signal is,

$$r(t) = A\,t \; ; \; t \geq 0 \qquad\qquad .....(2.5)$$
$$= 0 \; ; \; t < 0$$

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

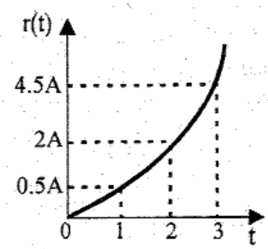


Fig 2.4 : Parabolic signal.

The mathematical representation of the parabolic signal is,

$$r(t) = \frac{At^2}{2} \; ; \; t \geq 0 \qquad\qquad .....(2.6)$$
$$= 0 \; ; \; t < 0$$

Note : Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called ***impulse signal.*** Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity.

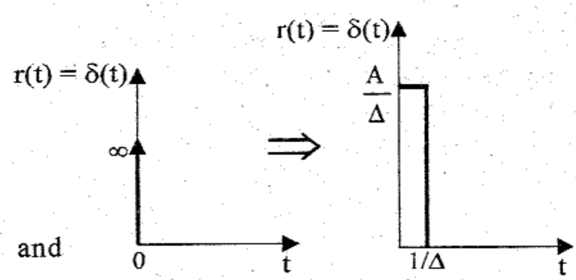


Fig 2.5 : Impulse signal.

The impulse signal is denoted by $\delta(t)$ and mathematically it is expressed as,

$$\delta(t) = \infty; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0; \; t \neq 0 \qquad\qquad .....(2.7)$$

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width but with area, A. Mathematically an impulse signal is the derivative of a step signal. Laplace transform of the impulse function is unity.

TABLE 2-1 : Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

2.3 IMPULSE RESPONSE

The response of the system, with input as impulse signal is called ***weighing function*** or ***impulse response*** of the system. It is also given by the inverse Laplace transform of the system transfer function, and denoted by m(t).

$$\text{Impulse response, } m(t) = \mathcal{L}^{-1}\{R(s)\,M(s)\} = \mathcal{L}^{-1}\{M(s)\} \qquad .....(2.8)$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)} \qquad \boxed{R(s) = 1, \text{ for impulse}}$$

Since impulse response (or weighing function) is obtained from the transfer function of the system, it shows the characteristics of the system. Also the response for any input can be obtained by convolution of input with impulse response.

2.4 ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n^{th} order differential equation shown in equation (2.9).

$$a_0\frac{d^n}{dt^n}p(t) + a_1\frac{d^{n-1}}{dt^{n-1}}p(t) + a_2\frac{d^{n-2}}{dt^{n-2}}p(t) + ...+ a_{n-1}\frac{d}{dt}p(t) + a_n\,p(t) = b_0\frac{d^m}{dt^m}q(t)$$
$$+ b_1\frac{d^{m-1}}{dt^{m-1}}q(t) + b_2\frac{d^{m-2}}{dt^{m-2}}q(t) + ...+ b_{m-1}\frac{d}{dt}q(t) + b_m\,q(t) \qquad .....(2.9)$$

where, $p(t) = \text{Output / Response}$; $q(t) = \text{Input / Excitation}$.

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called ***n^{th} order system.***

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in **s**, as shown in equation (2.10).

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots\dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_{n-1} s + a_n} \quad .....(2.10)$$

where, $P(s)$ = Numerator polynomial

$Q(s)$ = Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, $Q(s)$.

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots\dots\dots + a_{n-1} s + a_n$.

Now, **n** is the order of the system

When $n = 0$, the system is zero order system.

When $n = 1$, the system is first order system.

When $n = 2$, the system is second order system and so on.

Note : *The order can be specified for both open loop system and closed loop system.*

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\dots\dots(s+z_m)}{(s+p_1)(s+p_2)\dots\dots(s+p_n)} \quad .....(2.11)$$

where, $z_1, z_2, \dots\dots z_m$ are zeros of the system.

$p_1, p_2, \dots\dots p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

Note : *The zeros and poles are critical value, of s, at which the function T(s) attains extreme values 0 or ∞. When s takes the value of a zero, the function T(s) will be zero. When s takes the value of a pole, the function T(s) will be infinite.*

2.5 REVIEW OF PARTIAL FRACTION EXPANSION

The time response of the system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer function are encountered. They are,

Case 1 : Functions with separate poles.

Case 2 : Functions with multiple poles.

Case 3 : Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example.

Case 1 : When the transfer function has distinct poles

$$\text{Let, } \quad T(s) = \frac{K}{s(s+p_1)(s+p_2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A=T(s)\times s\big|_{s=0}\qquad B=T(s)\times(s+p_1)\big|_{s=-p_1}\qquad C=T(s)\times(s+p_2)\big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{2}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A=T(s)\times s\big|_{s=0}=\frac{2}{s(s+1)(s+2)}\times s\Big|_{s=0}=\frac{2}{(s+1)(s+2)}\Big|_{s=0}=\frac{2}{1\times 2}=1$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1.

$$B=T(s)\times(s+1)\big|_{s=-1}=\frac{2}{s(s+1)(s+2)}\times(s+1)\Big|_{s=-1}=\frac{2}{s(s+2)}\Big|_{s=-1}=\frac{2}{-1(-1+2)}=-2$$

C is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$C=T(s)\times(s+2)\big|_{s=-2}=\frac{2}{s(s+1)(s+2)}\times(s+2)\Big|_{s=-2}=\frac{2}{s(s+1)}\Big|_{s=-2}=\frac{2}{-2(-2+1)}=+1$$

$$\therefore T(s)=\frac{2}{s(s+1)(s+2)}=\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}$$

Case 2 : When the transfer function has multiple poles

$$\text{Let, } T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{(s+p_2)^2}+\frac{D}{(s+p_2)}$$

The residues A, B, C and D are given by,

$$A=T(s)\times s\big|_{s=0}\qquad B=T(s)\times(s+p_1)\big|_{s=-p_1}$$

$$C=T(s)\times(s+p_2)^2\big|_{s=-p_2}\qquad D=\frac{d}{ds}\left[T(s)\times(s+p_2)^2\right]\Big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+1)(s+2)^2}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)^2}\Big|_{s=0} = \frac{2}{1\times 2^2} = 0.5$$

B is obtained by multiplying T(s) by (s +1) and letting s = −1.

$$B = T(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)^2} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)^2}\Big|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

C is obtained by multiplying T(s) by $(s+2)^2$ and letting s = −2.

$$C = T(s) \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)}\Big|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

D is obtained by differentiating the product T(s) $(s+2)^2$ with respect to s and then letting s = −2.

$$D = \frac{d}{ds}\left[T(s) \times (s+2)^2\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{2}{s(s+1)}\right]\Big|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Big|_{s=-2} = \frac{-2(2(-2)+1)}{(-2)^2(-2+1)^2} = +1.5$$

$$\therefore\ T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}$$

Case 3 : When the transfer function has complex conjugate poles

$$\text{Let, } T(s) = \frac{K}{(s+p_1)(s^2+bs+c)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{(s+p_1)(s^2+bs+c)} = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c} \qquad \text{.....(2.12)}$$

The residue A is given by, $A = T(s) \times (s+p_1)\Big|_{s=-p_1}$

The residues B and C are solved by cross multiplying the equation (2.12) and then equating the coefficient of like power of **s.**

Finally express T(s) as shown below,

$$T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c}$$

$$\boxed{(x+y)^2 = x^2 + 2xy + y^2}$$

Let us express, $s^2 + bs$, in the form of $(x+y)^2$. This will require addition and subtraction of an extra term $(b/2)^2$.

$$\therefore\ T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2 + 2\times\frac{b}{2}s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2} = \frac{A}{s+p_1} + \frac{Bs+C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

$$= \frac{A}{s+p_1} + \frac{Bs}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} + \frac{C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

Example

$$\text{Let, } T(s)=\frac{1}{(s+2)\,(s^2+s+1)}$$

By partial fraction expansion,

$$T(s)=\frac{1}{(s+2)\,(s^2+s+1)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+s+1}$$

A is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$\therefore A=T(s)\times(s+2)\Big|_{s=-2}=\frac{1}{(s+2)\,(s^2+s+1)}\times(s+2)\Big|_{s=-2}=\frac{1}{(-2)^2-2+1}=\frac{1}{3}$$

To solve B and C, cross multiply the following equation and substitute the value of A. Then equate the like power of s.

$$\frac{1}{(s+2)\,(s^2+s+1)}=\frac{A}{s+2}+\frac{Bs+C}{s^2+s+1}$$

$$1=A(s^2+s+1)+(Bs+C)\,(s+2)$$

$$1=\frac{1}{3}(s^2+s+1)+Bs^2+2Bs+Cs+2C$$

$$1=\frac{s^2}{3}+\frac{s}{3}+\frac{1}{3}+Bs^2+2Bs+Cs+2C$$

$$\boxed{\begin{aligned}&s^2+s+1\\&=s^2+2\times\frac{s}{2}+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2\\&=\left(s+\frac{1}{2}\right)^2+\left(1-\frac{1}{4}\right)\\&=(s+0.5)^2+0.75\end{aligned}}$$

On equating the coefficient of s^2 terms, $0=\frac{1}{3}+B$; $\therefore B=-\frac{1}{3}$

On equating the coefficient of s terms, $0=\frac{1}{3}+2B+C$; $\therefore C=-\frac{1}{3}-2B=-\frac{1}{3}+\frac{2}{3}=\frac{1}{3}$

$$T(s)=\frac{\frac{1}{3}}{s}+\frac{\frac{-1}{3}s+\frac{1}{3}}{s^2+s+1}=\frac{1}{3s}-\frac{1}{3}\frac{s}{(s^2+s+1)}+\frac{1}{3}\frac{1}{(s^2+s+1)}$$

$$=\frac{1}{3s}-\frac{1}{3}\frac{s}{(s+0.5)^2+0.75}+\frac{1}{3}\frac{1}{(s+0.5)^2+0.75}$$

2.6 RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop order system with unity feedback is shown in fig 2.6.

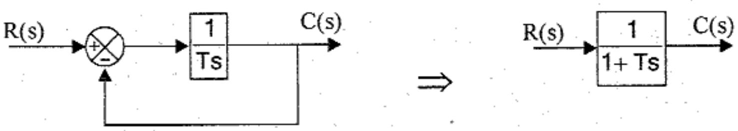


***Fig 2.6 :** Closed loop for first order system.*

The closed loop transfer function of first order system, $\frac{C(s)}{R(s)}=\frac{1}{1+Ts}$

If the input is unit step then, $r(t)=1$ and $R(s)=\frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s)=R(s)\frac{1}{(1+Ts)}=\frac{1}{s}\frac{1}{(1+Ts)}=\frac{1}{sT\left(\frac{1}{T}+s\right)}=\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$$

By partial fraction expansion,

$$C(s)=\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}=\frac{A}{s}+\frac{B}{\left(s+\frac{1}{T}\right)}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A=C(s)\times s\Big|_{s=0}=\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}\times s\Bigg|_{s=0}=\frac{\frac{1}{T}}{s+\frac{1}{T}}\Bigg|_{s=0}=\frac{\frac{1}{T}}{\frac{1}{T}}=1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = –1/T.

$$B=C(s)\times\left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}}=\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}\times\left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}}=\frac{\frac{1}{T}}{s}\Bigg|_{s=-\frac{1}{T}}=\frac{\frac{1}{T}}{\frac{-1}{T}}=-1$$

$$\therefore C(s)=\frac{1}{s}-\frac{1}{s+\frac{1}{T}}$$

$$\boxed{\mathcal{L}\left\{e^{-at}\right\}=\frac{1}{s+a}}$$

The response in time domain is given by,

$$c(t)=\mathcal{L}^{-1}\{C(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+\frac{1}{T}}\right\}=1-e^{-\frac{t}{T}} \quad .....(2.13)$$

The equation (2.13) is the response of the closed loop first order system for unit step input. For step input of step value, A, the equation (2.13) is multiplied by A.

$$\boxed{\begin{aligned}&\therefore \text{For closed loop first order system, Unit step response} = 1-e^{-\frac{t}{T}}\\ &\text{Step response} = A\left(1-e^{-\frac{t}{T}}\right)\end{aligned}}$$

When, $t = 0$, $c(t) = 1 - e^{0} = 0$

When, $t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

When, $t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

When, $t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

When, $t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$

When, $t = 5T$, $c(t) = 1 - e^{-5} = 0.993$

When, $t = \infty$, $c(t) = 1 - e^{-\infty} = 1$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state. The input and output signal of the first order system is shown in fig 2.7.

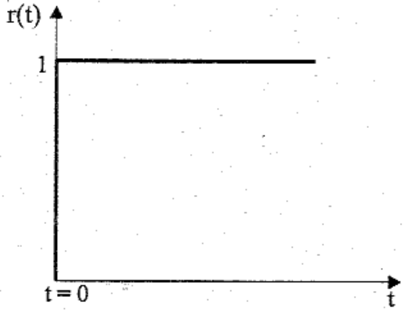


Fig 2.7a : Unit step input.

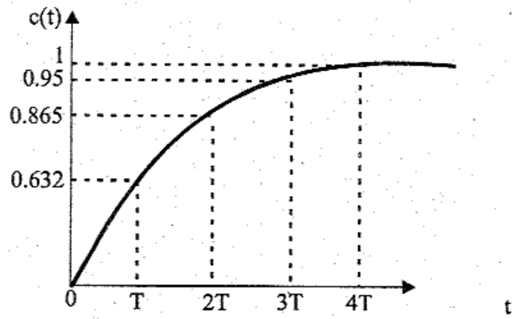


Fig 2.7b : Response for Unit step input.

Fig 2.7 : Response of first order system to Unit step input.

2.7 SECOND ORDER SYSTEM

The closed loop second order system is shown in fig 2.8

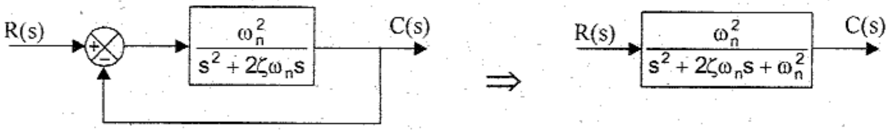


Fig 2.8 : Closed loop for second order system.

The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.14)$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

The ***damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Under damped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad .....(2.15)$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.16)$$

When $\zeta = 0,\ s_1, s_2 = \pm j\omega_n;\ \begin{cases}\text{roots are purely imaginary}\\ \text{and the system is undamped}\end{cases}$ .....(2.17)

When $\zeta = 1,\ s_1, s_2 = -\omega_n;\ \begin{cases}\text{roots are real and equal and}\\ \text{the system is critically damped}\end{cases}$ .....(2.18)

When $\zeta > 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1};\ \begin{cases}\text{roots are real and unequal and}\\ \text{the system is overdamped}\end{cases}$ .....(2.19)

When $0 < \zeta < 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\ \sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= -\zeta\omega_n \pm j\omega_d;\ \begin{cases}\text{roots are complex conjugate}\\ \text{the system is underdamped}\end{cases} \quad .....(2.20)$$

$$\text{where, } \omega_d = \omega_n\sqrt{1-\zeta^2} \quad .....(2.21)$$

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

2.7.1 RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, z = 0.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.22)$$

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\ \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\ \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.23)$$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\big|_{s=0} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times s\Bigg|_{s=0} = \frac{\omega_n^2}{s^2 + \omega_n^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s = j\omega_n$.

$$B = C(s) \times (s^2 + \omega_n^2)\big|_{s=j\omega} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2)\Bigg|_{s=j\omega} = \frac{\omega_n^2}{s}\Bigg|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$

$$\text{Time domain response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right\} = 1 - \cos \omega_n t \quad .....(2.24)$$

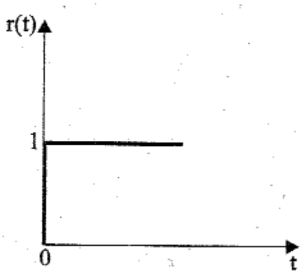


Fig 2.9.a : Input.

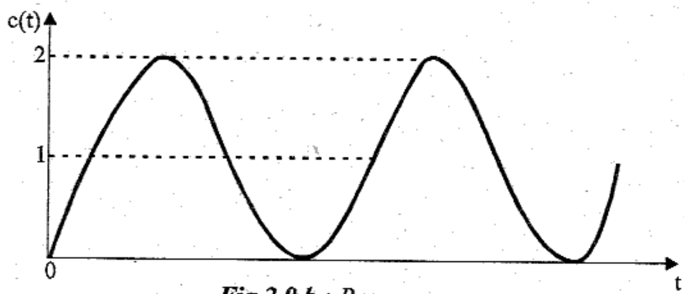


Fig 2.9.b : Response.

Fig 2.9 : Response of undamped second order system for unit step input.

Using equation (2.24), the response of undamped second order system for unit step input is sketched in fig 2.9, and observed that the response is completely oscillatory.

> ***Note :*** *Every practical system has some amount of damping. Hence undamped system does not exist in practice.*

The equation (2.24) is the response of undamped closed loop second order system for unit step input. For step input of step value A, the equation (2.24) should be multiplied by A.

$\therefore$ For closed loop undamped second order system,

Unit step response $= 1 - \cos \omega_n t$

Step response $= A(1 - \cos \omega_n t)$

2.7.2 RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Since $\zeta < 1$, ζ^2 is also less then 1, and so $1 - \zeta^2$ is always positive.

$$\therefore\ s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\,\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion, $C(s) = \dfrac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ .....(2.25)

A is obtained by multiplying C(s) by s and letting s = 0.

$$\therefore\ A = s\times C(s)\Big|_{s=0} = s\times\left.\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation (2.25) and equate like power of s.

On cross multiplication equation (2.25) after substituting A = 1, we get,

$$\omega_n^2 = s^2+2\zeta\omega_n s+\omega_n^2+(Bs+C)s$$

$$\omega_n^2 = s^2+2\zeta\omega_n s+\omega_n^2+Bs^2+Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B$ $\quad\therefore B = -1$

Equating coefficient of s we get, $0=2\zeta\omega_n + C$ $\quad\therefore C = -2\zeta\omega_n$

$$\therefore\ C(s) = \frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad .....(2.26)$$

Let us add and subtract $\zeta^2\omega_n^{\ 2}$ to the denominator of second term in the equation (2.26).

$$\therefore\ C(s) = \frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2+\zeta^2\omega_n^2-\zeta^2\omega_n^2} = \frac{1}{s}-\frac{s+2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2)+(\omega_n^2-\zeta^2\omega_n^2)}$$

$$= \frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)} = \frac{1}{s}-\frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \qquad \boxed{\omega_d=\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \qquad .....(2.27)$$

Let us multiply and divide by ω_d in the third term of the equation (2.27).

$$\therefore\ C(s) = \frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$
$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}$$

$$= 1-e^{-\zeta\omega_n t}\cos\omega_d t-\frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t = 1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad \boxed{\omega_d=\omega_n\sqrt{1-\zeta^2}}$$

$$= 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\ \cos\omega_d t+\zeta\ \sin\omega_d t\right) = 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\omega_d t\times\zeta+\cos\omega_d t\times\sqrt{1-\zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t\times\cos\theta+\cos\omega_d t\times\sin\theta)$$

$$= 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) \qquad .....(2.28)$$

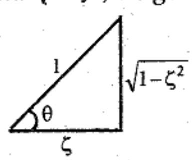
where, $\left(\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*

$\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$



The equation (2.28) is the response of under damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.28) should be multiplied by A.

∴ For closed loop under damped second order system,

$$\text{Unit step response} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \; ; \; \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step response} = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) \; ; \; \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

Using equation (2.28) the response of underdamped second order system for unit step input is sketched and observed that the response oscillates before settling to a final value. The oscillations depends on the value of damping ratio.

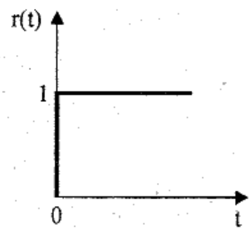


Fig 2.10.a : Input.

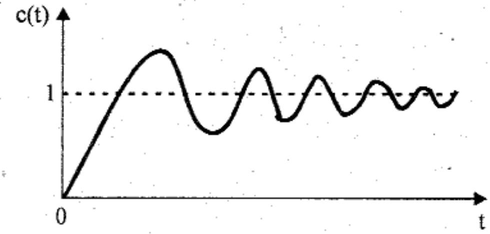


Fig 2.10.b : Response.

Fig 2.10 : Response of under damped second order system for unit step input.

2.7.3 RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad .....(2.29)$$

When input is unit step, $r(t) = 1$ and $R(s) = 1/s$.

∴ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s\,(s+\omega_n)^2} \qquad .....(2.30)$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)\Big|_{s=0} = \left.\frac{\omega_n^2}{(s+\omega_n)^2}\right|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 \times C(s)\Big|_{s=-\omega_n} = \left.\frac{\omega_n^2}{s}\right|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2 \times C(s)\right]\Big|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Big|_{s=-\omega_n} = \left.\frac{-\omega_n^2}{s^2}\right|_{s=-\omega_n} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$
$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \quad .....(2.31)$$

The equation (2.31) is the response of critically damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.31) should be multiplied by A.

$\therefore$ For closed loop critically damped second order system,

Unit step response $= 1 - e^{-\omega_n t}(1+\omega_n t)$

Step response $= A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right]$

Using equation (2.31), the response of critically damped second order system is sketched as shown in fig 2.11 and observed that the response has no oscillations.

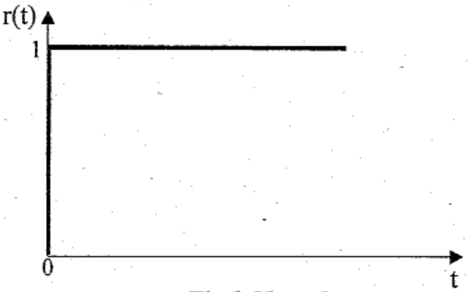


Fig 2.11.a : Input.

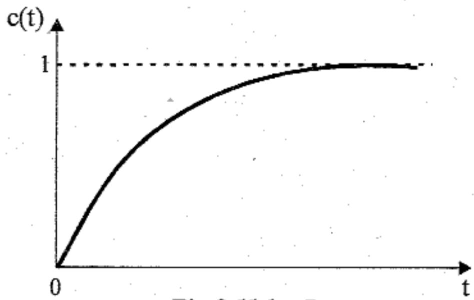


Fig 2.11.b : Response.

Fig 2.11 : Response of critically damped second order system for unit step input.

2.7.4 RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right] \quad .....(2.32)$$

Let $s_1 = -s_2$ and $s_2 = -s_b$ $\quad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ .....(2.33)

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \quad .....(2.34)$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)} \quad .....(2.35)$$

For unit step input r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)\,(s+s_2)} = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s\times C(s)\big|_{s=0} = s\times\frac{\omega_n^2}{s\,(s+s_1)\,(s+s_2)}\bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2\left(\zeta^2-1\right)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1)\times C(s)\big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\bigg|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_1}$$

$$C = C(s)\times(s+s_2)\big|_{s=-s_2} = \frac{\omega_n^2}{s\,(s+s_1)}\bigg|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_2}$$

The response in time domain, c(t) is given by,

$$c(t) = \mathcal{L}^{-1}\left\{=\frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_1}\;\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_2}\;\frac{1}{(s+s_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_1}\;e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\;\frac{1}{s_2}\;e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \quad .....(2.36$$

where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

The equation (2.36) is the response of overdamped closed loop system for unit step input. For step input of value, A, the equation (2.36) is multiplied by A.

∴ For closed loop over damped second order system,

Unit step response $= 1 - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}\;\dfrac{1}{s_1}\left(\dfrac{e^{-s_1 t}}{s_1} - \dfrac{e^{-s_2 t}}{s_2}\right)$ where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

Step response $= A\left[1 - \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}\;\dfrac{1}{s_1}\left(\dfrac{e^{-s_1 t}}{s_1} - \dfrac{e^{-s_2 t}}{s_2}\right)\right]$ $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

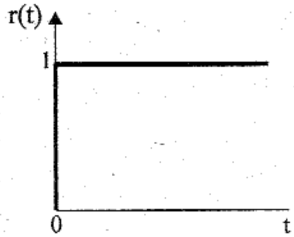


Fig 2.12.a : Input.

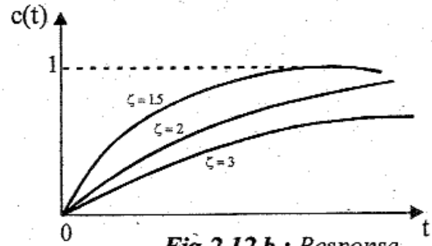


Fig 2.12.b : Response.

Fig 2.12 : Response of over damped second order system for unit step input.

Using equation (2.36), the response of overdamped second order system is sketched as shown in fig 2.12 and observed that the response has no oscillations but it takes longer time for the response to reach the final steady value.

2.8 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit-step input with various values of damping ratio is shown in fig 2.13.

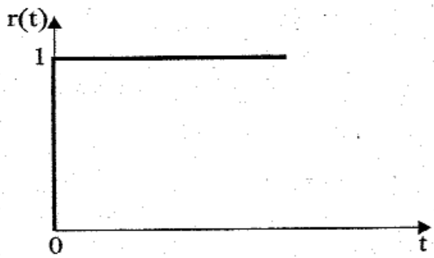


Fig 2.13.a : Input.

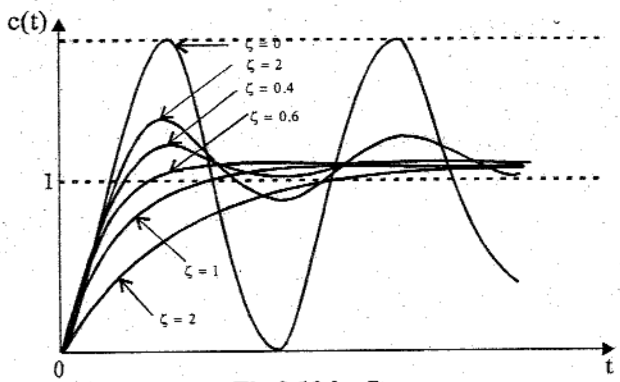


Fig 2.13.b : Response.

Fig 2.13 : Unit step response of second order system.

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t = 0$ will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in fig 2.14.

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

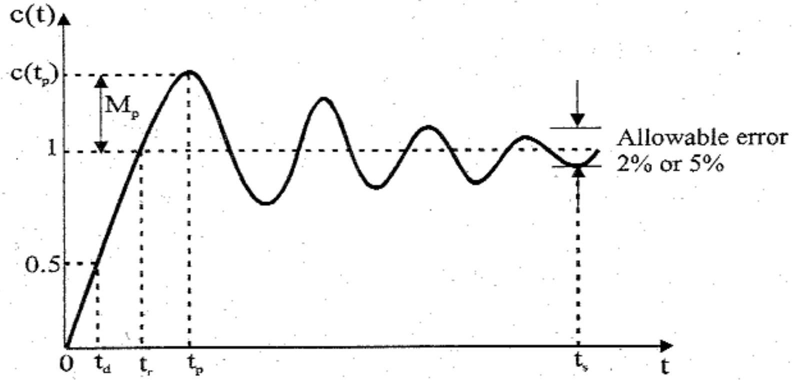


Fig 2.14 : Damped oscillatory response of second order system for unit step input.

The time domain specifications are defined as follows.

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

Let, $c(\infty)$ = Final value of c(t).

$c(t_p)$ = Maximum value of c(t).

$$\text{Now, Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \qquad .....(2.37)$$

$$\text{\% Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \qquad .....(2.38)$$

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

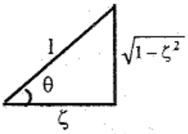
The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ **(Refer fig 2.14).**

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

> ***Note :*** *On constructing right angle triangle with* ζ *and* $\sqrt{1-\zeta^2}$, *we get*
>
> $\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$
>
> 
>

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When, $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\sin\phi = 0$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{\therefore \text{ Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}} \qquad \text{.....(2.39)}$$

Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ ***(refer note)***

$$\boxed{\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}} \qquad \text{.....(2.40)}$$

> ***Note :*** θ *or* $\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ *should be measured in radians.*

Peak time (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

$$\text{i.e., } \frac{d}{dt}c(t)\Big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to **t**.

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (-\zeta\omega_n) \sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right) \cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (\zeta\omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2} \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right] \quad \textit{(refer note)}$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

$$=\frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_d t)$$

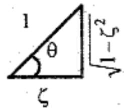
at $t = t_p$, $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_p}\sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi$, $\sin\phi = 0$

$\therefore\ \omega_d t_p = \pi$

> ***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $\sin\theta = \sqrt{1-\zeta^2}$
>
> $\cos\theta = \zeta$
>
> 
>

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}} \quad .....(2.41)$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \quad .....(2.42)$$

Peak overshoot (M_p)

$$\boxed{\%\text{Peak overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100} \quad .....(2.43)$$

where, $c(t_p)$ = Peak response at $t = t_p$.

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

At $t=\infty$, $\quad c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1-0 = 1$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

At $t = t_p$, $\quad c(t) = c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$

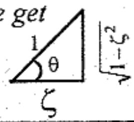
$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$\boxed{\sin(\pi+\theta) = -\sin\theta}$$

$$= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\theta)}{\sqrt{1-\zeta^2}}$$

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\theta$$

> ***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $\sin\theta = \sqrt{1-\zeta^2}$
>
> 
>

$$= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \quad .....(2.44)$$

$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100 = \frac{1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}-1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\boxed{\therefore \text{ Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100} \quad .....(2.45)$$

Settling time (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \dfrac{1}{\zeta\omega_n}$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \quad \text{(for 2\% error)}} \quad .....(2.46)$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \quad \text{(for 5\% error)}} \quad .....(2.47)$$

In general for a specified percentage error, Settling time can be evaluted using equation (2.48).

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{ln(\%\text{ error})}{\zeta\omega_n} = \frac{ln(\%\text{ error})}{T}} \quad .....(2.48)$$

EXAMPLE 2.1

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \dfrac{4}{s(s+5)}$ and when the input is unit step.

SOLUTION

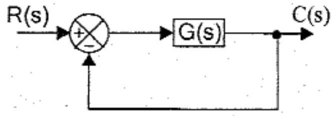


Fig 1 : Closed loop system.

The closed loop system is shown in fig 1.

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{4}{s\,(s+5)}}{1+\dfrac{4}{s\,(s+5)}} = \frac{\dfrac{4}{s\,(s+5)}}{\dfrac{s\,(s+5)+4}{s\,(s+5)}} = \frac{4}{s\,(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)\,(s+1)}$$

The response in s-domain, $C(s) = R(s)\ \dfrac{4}{(s+1)\,(s+4)}$

Since the input is unit step, $R(s) = \dfrac{1}{s}$; $\quad \therefore C(s) = \dfrac{4}{s\,(s+1)\,(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s) \times s\big|_{s=0} = \left.\frac{4}{(s+1)\,(s+4)}\right|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s) \times (s+1)\big|_{s=-1} = \left.\frac{4}{s\,(s+4)}\right|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s) \times (s+4)\big|_{s=-4} = \left.\frac{4}{s\,(s+1)}\right|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s} - \dfrac{4}{3}\,\dfrac{1}{s+1} + \dfrac{1}{3}\,\dfrac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

RESULT

Response of unity feedback system, $c(t) = 1 - \dfrac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

EXAMPLE 2.2

A positional control system with velocity feedback is shown in fig 1. What is the response of the system for unit step input.

SOLUTION

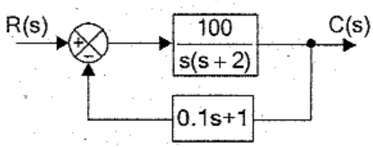


Fig 1 : Positional control system.

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

Given that, $G(s) = \dfrac{100}{s(s+2)}$ and $H(s) = 0.1s+1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{100}{s\,(s+2)}}{1+\left(\dfrac{100}{s\,(s+2)}\right)(0.1s+1)} = \frac{\dfrac{100}{s\,(s+2)}}{\dfrac{s(s+2)+100\,(0.1s+1)}{s\,(s+2)}} = \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

Here $(s^2 + 12s + 100)$ is characteristic polynomial. The roots of the characteristic polynomial are,

$$s_1, s_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

The roots are complex conjugate. The system is underdamped and so the response of the system will have damped oscillations.

$$\text{The response in s-domain, } C(s) = R(s)\,\frac{100}{s^2+12s+100}$$

$$\text{Since input is unit step, } R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s}\;\frac{100}{s^2+12s+100} = \frac{100}{s(s^2+12s+100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \left.\frac{100}{s^2+12s+100}\right|_{s=0} = \frac{100}{100} = 1$$

The residue B and C are evaluated by cross multiplying the following equation and equating the coefficients of like power of s.

$$\frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$100 = A(s^2+12s+100) + (Bs+C)\,s$$

$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\therefore B = -A = -1$

On equating coefficients of s we get, $0 = 12A + C$ $\therefore C = -12A = -12$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100} = \frac{1}{s} - \frac{s+12}{s^2+12s+36+64} = \frac{1}{s} - \frac{s+6+6}{(s+6)^2+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2} = \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

$$\text{Time response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}\right\}$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

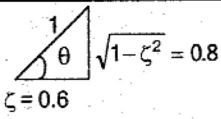
The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$. The damping ratio ζ is evaluated by comparing the closed loop transfer function of the system with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{100}{s^2+12s+100}$$

On comparing we get, $\omega_n^2 = 100$ | $2\zeta\omega_n = 12$

$\therefore \omega_n = 10$ | $\therefore \zeta = \dfrac{12}{2\omega_n} = \dfrac{12}{2 \times 10} = 0.6$

Constructing right angled triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$\cos\theta = 0.6$

$\sin\theta = 0.8$

$$\sin\theta = 0.8 \ ; \ \cos\theta = 0.6 \ ; \ \tan\theta = \frac{0.8}{0.6}$$

$$\therefore \theta = \tan^{-1}\frac{0.8}{0.6} = 53° = 53° \times \frac{\pi}{180°}\text{rad} = 0.925 \text{ rad.}$$

$$\therefore \text{ Time response, } c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] = 1 - e^{-6t}\frac{10}{8}\left[\frac{6}{10}\sin 8t + \frac{8}{10}\cos 8t\right]$$

$$= 1 - \frac{10}{8}e^{-6t}[\sin 8t \times 0.6 + \cos 8t \times 0.8] = 1 - 1.25\, e^{-6t}[\sin 8t \cos\theta + \cos 8t \sin\theta]$$

$$= 1 - 1.25\, e^{-6t}[\sin\,(8t+\theta)] = 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

Note : θ is expressed in radians

RESULT

The response in time domain,

$$c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] \quad \text{or} \quad c(t) = 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

EXAMPLE 2.3

The response of a servomechanism is, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOLUTION

Given that, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - 1.2\frac{1}{(s+10)} = \frac{(s+60)\,(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 1.2s^2 - 72s}{s\,(s+60)\,(s+10)} = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\,\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, R(s) = 1/s.

$$\therefore C(s) = R(s)\frac{60}{(s+60)\,(s+10)} = R(s)\frac{600}{s^2 + 70s + 600}$$

$$\therefore \text{ The closed loop transfer function of the system, } \frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore \zeta = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.49} = 1.43$$

RESULT

The closed loop transfer function of the system, $\dfrac{C(s)}{R(s)} = \dfrac{600}{s^2 + 70s + 600}$

Natural frequency of oscillation, $\omega_n = 24.49$ rad/sec

Damping ratio, $\zeta = 1.43$

EXAMPLE 2.4

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

SOLUTION

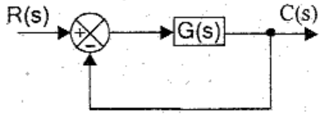


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (s +10)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K=100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

EXAMPLE 2.5

The open loop transfer function of a unity feedback system is given by G(s) = K/s (sT + 1), where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOLUTION

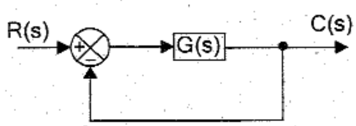


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (sT +1)

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s(sT+1)}{1+K/s(sT+1)} = \frac{K}{s(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

On comparing we get,

$$\omega_n^2 = K/T \qquad \therefore \omega_n = \sqrt{K/T}$$

$$2\zeta\omega_n = 1/T \qquad \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $\ln M_p = \dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $(\ln M_p)^2 = \dfrac{\zeta^2\pi^2}{1-\zeta^2}$

On crossing multiplication we get,

$$(1-\zeta^2)(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 - \zeta^2(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 = \zeta^2\pi^2 + \zeta^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \zeta^2\left[\pi^2+(\ln M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2} \qquad ....(1)$$

$$\text{But } \zeta = \frac{1}{2\sqrt{KT}}, \quad \therefore \zeta^2 = \frac{1}{4KT} \qquad ....(2)$$

On equating, equation (1) & (2) we get,

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$K = \frac{\pi^2+(\ln M_p)^2}{4T(\ln M_p)^2}$$

When, $K = K_1$, $M_p = 0.75$, $\therefore K_1 = \dfrac{\pi^2+(\ln 0.75)^2}{4T(\ln 0.75)^2} = \dfrac{9.952}{0.331T} = \dfrac{30.06}{T}$

When, $K = K_2$, $M_p = 0.25$, $\therefore K_2 = \dfrac{\pi^2+(\ln 0.25)^2}{4T(\ln 0.25)^2} = \dfrac{11.79}{7.68T} = \dfrac{1.53}{T}$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\,30.06}{(1/T)\,1.53} = 19.6$$

$$K_1 = 19.6\,K_2 \quad \text{(or)} \quad K_2 = \frac{1}{19.6}K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

RESULT

The value of gain, K should be reduced approximately 20 times to reduce peak overshoot from 0.75 to 0.25.

EXAMPLE 2.6

A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

SOLUTION

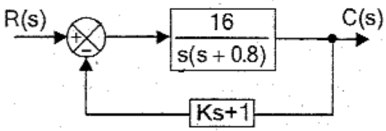


Fig 1

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)\,H(s)}$

Given that G(s) = 16/s(s + 0.8) and H(s) = Ks +1

$$\therefore \frac{C(s)}{R(s)} = \frac{\dfrac{16}{s\,(s+0.8)}}{1+\dfrac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8)+16(Ks+1)}$$

$$= \frac{16}{s^2+0.8s+16Ks+16} = \frac{16}{s^2+(0.8+16K)\,s+16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{16}{s^2+(0.8+16K)\,s+16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad\qquad 0.8+16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad/sec} \qquad\qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2\times 0.5\times 4-0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2+(0.8+16\times 0.2)\,s+16} = \frac{16}{s^2+4s+16}$$

Given that the damping ratio, ζ= 0.5. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s - domain, $C(s) = R(s)\,\dfrac{16}{s^2+4s+16}$

For unit step input, R(s) = 1/s.

$$\therefore C(s) = \frac{1}{s}\,\frac{16}{s^2+4s+16} = \frac{16}{s\,(s^2+4s+16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s)\times s\big|_{s=0} = \left.\frac{16}{s^2+4s+16}\right|_{s=0} = \frac{16}{16} = 1$$

The residues B and C are evaluated by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{16}{s\,(s^2+4s+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

On cross multiplication we get, $16 = A\,(s^2+4s+16) + (Bs+C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B \quad \therefore\ B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C \ \therefore\ C = -4A = -4$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

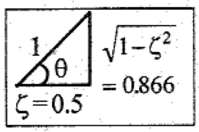
$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\ e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$.

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$$\sin\theta = 0.866 = \sqrt{3}/2\ ; \quad \cos\theta = 0.5 = 1/2\ ; \ \tan\theta = 1.732$$

$$\therefore\ \theta = \tan^{-1}1.732 = 60° = 1.047 \text{ rad}$$

$\therefore$ The response in time domain,

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \times 2 \times \sin\sqrt{12}\,t \times \frac{1}{2} + \frac{2}{\sqrt{3}} \times \cos\sqrt{12}\,t \times \frac{\sqrt{3}}{2}\right]$$

$$= 1 - e^{-2t}\,\frac{2}{\sqrt{3}}\left[\sin\sqrt{12}\,t\cos\theta + \cos\sqrt{12}\,t\sin\theta\right]$$

Note : θ is expressed in radians.

$$= 1 - \frac{2}{\sqrt{3}}\ e^{-2t}\left[\sin(\sqrt{12}\,t + \theta)\right] = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$$

Damped frequency of oscillation $\Big\}\ \omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464$ rad / sec

$$\therefore\ \text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

% Maximum overshoot $\Big\}\ \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

RESULT

The time domain response, $c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$

$$(\text{or}) \quad c(t) = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$$

Rise time, $t_r = 0.6046$ sec

Peak time, $t_p = 0.907$ sec

% Maximum overshoot, $\%M_p = 16.3\%$

Settling time, $t_s = 1.5$ sec, for 5% error

$= 2$ sec, for 2% error

EXAMPLE 2.7

A unity feedback control system is characterized by the following open loop transfer function $G(s) = (0.4\,s+1)/s(s+0.6)$. Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)\,H(s)}$

Given that, $G(s) = (0.4\,s+1)/s(s+0.6)$

For unity feedback system, $H(s) = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{0.4\,s+1}{s(s+0.6)}}{1+\dfrac{0.4\,s+1}{s(s+0.6)}} = \frac{0.4\,s+1}{s\,(s+0.6)+0.4\,s+1}$$

$$= \frac{0.4\,s+1}{s^2+0.6s+0.4s+1} = \frac{0.4\,s+1}{s^2+s+1}$$

The s - domain response, $C(s) = R(s) \times \dfrac{0.4\,s+1}{s^2+s+1}$

For step input, $R(s) = 1/s$.

$$\therefore\ C(s) = \frac{1}{s}\,\frac{0.4\,s+1}{s^2+s+1} = \frac{0.4\,s+1}{s(s^2+s+1)}$$

By partial fraction expansion C(s) can be expressed as,

$$C(s) = \frac{0.4\,s+1}{s\,(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

The residue A is solved by multiplying C(s) by s and letting $s = 0$.

$$\therefore\ A = C(s) \times s\Big|_{s=0} = \left.\frac{0.4\,s+1}{s^2+s+1}\right|_{s=0} = 1$$

The residues B and C are solved by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{0.4\,s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

On cross multiplication we get,

$$0.4\,s+1 = A(s^2+s+1) + (Bs+C)\,s$$

$$0.4\,s+1 = As^2 + As + A + Bs^2 + Cs$$

On equating coefficients of s^2 we get, $0 = A + B$ $\quad \therefore B = -A = -1$

On equating coefficients of s we get, $0.4 = A + C$ $\quad \therefore C = 0.4 - A = -0.6$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-0.6}{s^2+s+1} = \frac{1}{s} - \frac{s+0.6}{s^2+s+0.25+0.75} = \frac{1}{s} - \frac{s+0.6}{\left(s^2+2\times 0.5s+0.5^2\right)+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5+0.1}{(s+0.5)^2+0.75} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\;\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

∴ The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right\}$$

$$= 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}\,e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

$$\text{The transient response of } c(t) = e^{-0.5t}\left[0.1155\,\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$$

The value of ζ and ω_n can be estimated by comparing the characteristic equation of the system with standard form of second order characteristic equation.

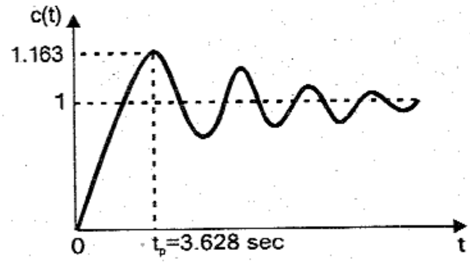


Fig 1 : Response of under damped system.

$$\therefore s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + s + 1$$

On comparing we get,

$$\omega_n^2 = 1 \qquad\qquad 2\zeta\omega_n = 1$$

$$\therefore \omega_n = 1 \text{ rad/sec} \qquad\qquad \therefore \zeta = \frac{1}{2\omega_n} = \frac{1}{2} = 0.5$$

$$\text{Maximum overshoot, } M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$$

$$\%\text{ Maximum overshoot, } \%M_p = M_p \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628 \text{ sec}$$

The response of the system is underdamped and it is shown in fig 1.

RESULT

Transient response of the system, c(t) = $e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

% Maximum peak overshoot, $\%M_p$ = 16.3 %

Peak time, t_p = 3.628 sec

EXAMPLE 2.8

A unity feedback control system has an amplifier with gain K_A = 10 and gain ratio, G(s) = 1/s(s +2) in the feed forward path. A derivative feedback, H(s) = sK_o is introduced as a minor loop around G(s). Determine the derivative feedback constant, K_o so that the system damping factor is 0.6.

SOLUTION

The given system can be represented by the block diagram shown in fig 1.

$$\text{Here, } K_A = 10\,;\; G(s) = \frac{1}{s(s+2)} \quad \text{and} \quad H(s) = sK_o$$

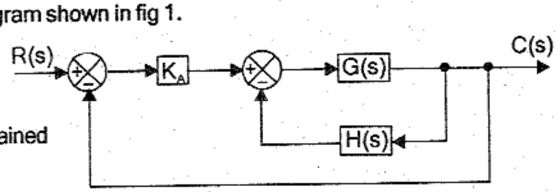
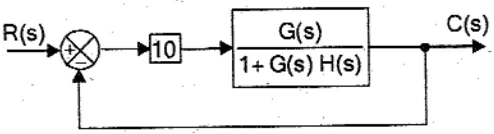


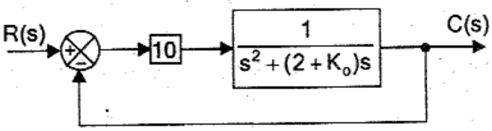
Fig 1.

The closed loop transfer function of the system can be obtained by block diagram reduction techniques.

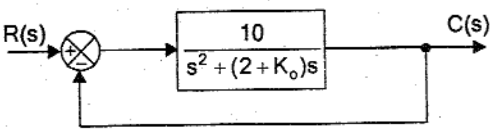
Step 1: Reducing the inner feedback loop.



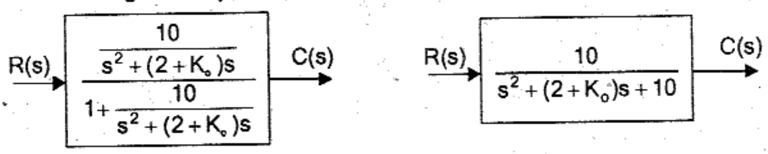
$$\frac{G(s)}{1+G(s)\,H(s)} = \frac{\frac{1}{s\,(s+2)}}{1+\frac{1}{s\,(s+2)}\,sK_o} = \frac{1}{s\,(s+2)+sK_o} = \frac{1}{s^2+2s+sK_o} = \frac{1}{s^2+(2+K_o)\,s}$$



Step 2 : Combining blocks in cascade



Step 3 : Reducing the unity feedback path



The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{10}{s^2+(2+K_o)\,s+10}$ .....(1)

The given system is a second order system. The value of K_o can be determined by comparing the system transfer function with standard form of second order transfer function given below.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad .....(2)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10$$

$$\therefore\ \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2+K_o = 2\zeta\omega_n$$

$$\therefore\ K_o = 2\zeta\omega_n - 2$$

$$= 2 \times 0.6 \times 3.162 - 2 = 1.7944$$

RESULT

The value of constant, $K_o = 1.7944$

EXAMPLE 2.9

A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

SOLUTION

Note : The formulae for rise time, percentage overshoot and peak time remains same for unit step and step input.

The unity feedback system is shown in fig 1.

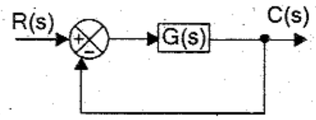


Fig 1 : Unity feedback system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10} \quad \text{..... (1)}$$

The values of damping ratio ζ and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad \text{.....(2)}$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10 \qquad\qquad 2\zeta\omega_n = 2$$

$$\therefore \omega_n = \sqrt{10} = 3.162 \text{ rad / sec} \qquad \therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot } = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time, } \quad t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

RESULT

Rise time, t_r	=	0.63 sec
Percentage overshoot, $\%M_p$	=	35.12%
Peak overshoot	=	4.2144 units, (for a input of 12 units)
Peak time, t_p	=	1.047 sec
Settling time, t_s	=	3 sec for 5% error
	=	4 sec for 2% error

EXAMPLE 2.10

A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\ e$

Where c is the displacement of the output shaft, r is the displacement of the input shaft and e = r − c. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

SOLUTION

The mathematical equations governing the system are,

$$\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\ e \quad .....(1)$$

$$e = r - c \quad .....(2)$$

Put $e = r - c$ in equation (1),

$$\therefore\ \frac{d^2c}{dt^2}+8\frac{dc}{dt}=64(r-c) \quad .....(3)$$

Let $\mathcal{L}\{c\} = C(s)$ and $\mathcal{L}\{r\} = R(s)$

On taking Laplace transform of equation (3) we get,

$$s^2\,C(s) + 8s\,C(s) = 64\,[R(s) - C(s)]$$

$$\therefore\ s^2\,C(s) + 8s\,C(s) + 64\,C(s) = 64\,R(s)$$

$$(s^2 + 8s + 64)\,C(s) = 64\,R(s)$$

$$\therefore\ \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64} \quad .....(4)$$

The ratio C(s)/R(s) is the closed loop transfer function of the system. On comparing the system transfer function with standard form of second order transfer function, we can estimate the values of ζ and ω_n.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\}\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad .....(5)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 64 \qquad\qquad 2\zeta\omega_n = 8$$

$$\therefore\ \omega_n = 8\ \text{rad / sec} \qquad\qquad \zeta = \frac{8}{2\omega_n} = \frac{8}{2\times 8} = 0.5$$

Percentage peak overshoot, $\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}\times 100 = 16.3\%$

RESULT

Undamped natural frequency of oscillation, ω_n = 8 rad/sec

Damping ratio, ζ = 0.5

Percentage peak overshoot, $\%M_p$ = 16.3%

2.9 TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if **N** is the number of poles at the origin then the type number is **N.**

The loop transfer function can be expressed as a ratio of two polynomials in **s.**

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\text{..........}}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\text{......}} \quad \text{.....(2.49)}$$

where, z_1, z_2, z_3, ... are zeros of transfer function

p_1, p_2, p_3, ... are poles of transfer function

K = Constant

N = Number of poles at the origin

The value of **N** in the denominator polynomial of loop transfer function shown in equation (2.49) decides the type number of the system.

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

2.10 STEADY STATE ERROR

The steady state error is the value of error signal e(t), when **t** tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig 2.15.

Let, R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

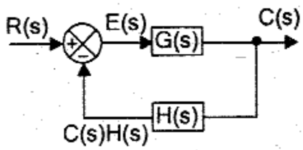


Fig 2.15.

The error signal, E(s) = R(s) – C(s) H(s) .....(2.50)

The output signal, C(s) = E(s) G(s) .....(2.51)

On substituting for C(s) from equation (2.51) in equation (2.50) we get,

$$E(s) = R(s) - [E(s)\ G(s)]\ H(s)$$

$$E(s) + E(s)\ G(s)\ H(s) = R(s)$$

$$E(s)\ [1 + G(s)\ H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)\,H(s)} \quad \text{.....(2.52)}$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\} \quad .....(2.53)$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) \quad .....(2.54)$$

The final value theorem of Laplace transform states that,

$$\textbf{If,}\quad \mathbf{F(s) = \mathcal{L}\{f(t)\}}\ \textbf{then,}\quad \underset{t\to\infty}{\mathbf{Lt}}\ \mathbf{f(t)} = \underset{s\to 0}{\mathbf{Lt}}\ \mathbf{s\,F(s)} \quad .....(2.55)$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\; e(t) = \underset{s\to 0}{\mathrm{Lt}}\; sE(s) = \underset{s\to 0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)} \quad .....(2.56)$$

2.11 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

$$\text{Positional error constant,}\quad K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s) \quad .....(2.57)$$

$$\text{Velocity error constant,}\quad K_v = \underset{s\to 0}{\mathrm{Lt}}\; s\,G(s)\,H(s) \quad .....(2.58)$$

$$\text{Acceleration error constant,}\quad K_a = \underset{s\to 0}{\mathrm{Lt}}\; s^2 G(s)\,H(s) \quad .....(2.59)$$

The K_p, K_v and K_a are in general called static error constants.

2.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{sR(s)}{1+G(s)\,H(s)}$$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s)} = \frac{1}{1+K_p} \quad .....(2.60)$$

$$\text{where, } K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s)$$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s\to 0}{\mathrm{Lt}}\; G(s)\,H(s) = \underset{s\to 0}{\mathrm{Lt}}\; K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s) = \underset{s\to 0}{Lt}\; K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

2.13 STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt}\; \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit ramp, $R(s) = \dfrac{1}{s^2}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\; \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\; sG(s)\,H(s)} = \frac{1}{K_v} \qquad .....(2.61)$$

where, $K_v = \underset{s\to 0}{Lt}\; s\,G(s)\,H(s)$

The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

2.14 STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s \to 0}{Lt} \dfrac{sR(s)}{1 + G(s)\,H(s)}$

When the input is unit parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore e_{ss} = \underset{s \to 0}{Lt} \frac{s\frac{1}{s^3}}{1 + G(s)\,H(s)} = \underset{s \to 0}{Lt} \frac{1}{s^2 + s^2\,G(s)\,H(s)} = \frac{1}{\underset{s \to 0}{Lt}\; s^2 G(s)\,H(s)} = \frac{1}{K_a} \qquad .....(2.62)$$

where, $K_a = \underset{s \to 0}{Lt}\; s^2 G(s)\,H(s)$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s \to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s \to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s \to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K \frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s \to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s \to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

2.15 GENERALIZED ERROR COEFFICIENT

The drawback in static error coefficients is that it does not show the variation of error with time and input should be a standard input. The generalized error coefficients gives the steady state error as a function of time. Also using the generalized error coefficients, the steady state error can be found for any type of input.

The error signal in s-domain, E(s) can be expressed as a product of two s-domain functions.

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{1}{1+G(s)\,H(s)} R(s) = F(s)\,R(s) \qquad .....(2.63)$$

$$\text{where, } F(s) = \frac{1}{1+G(s)\,H(s)}$$

Let, $e(t) = \mathcal{L}^{-1}\{E(s)\}$ (error signal in time domain)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$r(t) = \mathcal{L}^{-1}\{R(s)\}$ (input signal in time domain)

The convolution theorem of Laplace transform states that the Laplace transform of the convolution of two time domain signals is equal to the product of their individual Laplace transform.

i.e., $\mathcal{L}\{f(t) * r(t)\} = F(s)\,R(s)$

where $*$ is the symbol for convolution operation

$$\therefore \mathcal{L}^{-1}\{F(s)\,R(s)\} = f(t) * r(t) \qquad .....(2.64)$$

From equation (2.63) & (2.64) we can write,

$e(t) = f(t) * r(t)$

Mathematically the convolution of $f(t)$ and $r(t)$ is defined as,

$$f(t) * r(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT \ ; \quad \text{where T is a dummy variable}$$

$$\therefore \ e(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT$$

It is assumed that the input signal starts only at t = 0 and does not exist before t = 0. Also we are interested in finding error signal at any time t after t = 0 (i.e., for t > 0). Hence in the above equation the limit of integral can be changed as 0 to t.

$$\therefore e(t) = \int_{0}^{t} f(T)\,r(t-T)\,dT$$

Using Taylor's series expansion the signal r(t–T) can be expressed as,

$$r(t-T) = r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\,\ddot{r}(t) - \frac{T^3}{3!}\,\dddot{r}(t) + \ldots\ldots + (-1)^n\,\frac{T^n}{n!}\,\overset{n.}{r}(t)\ldots$$

where, $\dot{r}(t) = 1^{st}$ derivative of r(t)

$\ddot{r}(t) = 2^{nd}$ derivative of r(t)

$\vdots$

$\overset{n.}{r}(t) = n^{th}$ derivative of r(t)

On substituting the Taylor's series expansion of r(t – T), the error e(t) can be written as,

$$e(t) = \int_0^t f(T)\left[r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\,\ddot{r}(t) - \frac{T^3}{3!}\,\dddot{r}(t) + \ldots + (-1)^n\frac{T^n}{n!}\,\overset{n.}{r}(t)\ldots\right]dT$$

$$e(t) = \int_0^t f(T)\,r(t)\,dT - \int_0^t f(T)\,T\,\dot{r}(t)\,dT + \int_0^t f(T)\frac{T^2}{2!}\,\ddot{r}(t)\,dT$$

$$-\int_0^t f(T)\frac{T^3}{3!}\,\dddot{r}(t) + \ldots + \int_0^t f(T)\,(-1)^n\,\frac{T^n}{n!}\,\overset{n.}{r}(t)\,dT\ldots\infty$$

Since $r(t), \dot{r}(t), \ddot{r}(t), \ldots\ldots \overset{n.}{r}(t)$ are constants when the integration is done with respect to T, the error signal can be written as,

$$e(t) = r(t)\int_0^t f(T)\,dT - \dot{r}(t)\int_0^t Tf(T)\,dt + \frac{\ddot{r}(t)}{2!}\int_0^t T^2 f(T)\,dt$$

$$-\frac{\dddot{r}(t)}{3!}\int_0^t T^3 f(T)\,dt + \ldots + (-1)^n\,\frac{\overset{n.}{r}(t)}{n!}\int_0^t T^n\,f(T)\,dt\ldots$$

Let, $C_0 = +\int_0^t f(T)\,dT$ $\qquad C_3 = -\int_0^t T^3 f(T)\,dT$

$C_1 = -\int_0^t Tf(T)\,dT$ $\qquad \vdots$

$C_2 = +\int_0^t T^2 f(T)\,dT$ $\qquad C_n = (-1)^n\int_0^t T^n f(T)\,dT$

$$e(t) = r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + \ldots + \overset{n.}{r}(t)\,\frac{C_n}{n!} + \ldots\ldots$$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\,\ddot{r}(t) + \frac{C_3}{3!}\,\dddot{r}(t) + \ldots + \frac{C_n}{n!}\,\overset{n.}{r}(t)\ldots\ldots \qquad \ldots\ldots(2.65)$$

The equation (2.65) is the general equation for error signal, e(t).

The coefficients C_0, C_1, C_2 ,......C_n are called the generalized error coefficients or dynamic error coefficients.

The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t).

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\left[r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + \ldots + \overset{n.}{r}(t)\,\frac{C_n}{n!} + \ldots\ldots\right]$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\,\ddot{r}(t) + \frac{C_3}{3!}\,\dddot{r}(t) + \ldots + \frac{C_n}{n!}\,\overset{n.}{r}(t)\ldots\ldots \qquad \ldots\ldots(2.66)$$

2.16 EVALUATION OF GENERALIZED ERROR COEFFICIENTS

The generalized error coefficient is given by,

$$C_n = (-1)^n \int_0^t T^n f(T)\, dT\,; \qquad \text{where } F(s) = \frac{1}{1+G(s)\,H(s)}$$

We know that $\mathcal{L}\{f(T)\} = F(s)$, hence by the definition of Laplace transform,

$$F(s) = \int_0^t f(T)\, e^{-sT} dT \qquad .....(2.67)$$

On taking $\underset{s\to 0}{\mathrm{Lt}}$ on both sides of equation (2.67) we get,

$$\underset{s\to 0}{\mathrm{Lt}}\; F(s) = \underset{s\to 0}{\mathrm{Lt}} \int_0^t f(T)\, e^{-sT} dT$$

$$= \int_0^t f(T)\; \underset{s\to 0}{\mathrm{Lt}}\; e^{-sT} dT = \int_0^t f(T)\, dT = C_0$$

$$\therefore \boxed{\mathbf{C_0 = \underset{s\to 0}{Lt}\, F(s)}} \qquad .....(2.68)$$

On differentiating equation (2.68) with respect to s we get,

$$\frac{d}{ds} F(s) = \frac{d}{ds}\int_0^t f(T)\, e^{-sT}\, dT$$

$$= \int_0^t f(T)\, \frac{d}{ds}\left(e^{-sT}\right) dT = \int_0^t f(T)\, (-T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\, e^{-sT} dT \qquad .....(2.69)$$

On taking $\underset{s\to 0}{\mathrm{Lt}}$ on both sides of equation (2.69) we get,

$$\underset{s\to 0}{\mathrm{Lt}}\; \frac{d}{ds} F(s) = \underset{s\to 0}{\mathrm{Lt}}\; -\int_0^t Tf(T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\; \underset{s\to 0}{\mathrm{Lt}}\; e^{-sT} dT = -\int_0^t Tf(T)\, dT = C_1$$

$$\therefore \boxed{\mathbf{C_1 = \underset{s\to 0}{Lt}\, \frac{d}{ds} F(s)}} \qquad .....(2.70)$$

On differentiating equation (2.68) on both sides with respect to s we get,

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^t Tf(T)\, e^{-sT}\, dT\right]$$

$$\frac{d^2}{ds^2} F(s) = \left[-\int_0^t Tf(T)\, \frac{d}{ds}\left(e^{-sT}\right) dT\right] = -\int_0^t Tf(T)\, (-T)\, e^{-sT} dT$$

$$\frac{d^2\,(F(s))}{ds^2} = \int_0^t T^2 f(T)\, e^{-sT} dT \qquad .....(2.71)$$

Applying the limit $s \to 0$ on both sides of the equation (2.71) we get,

$$\underset{s\to 0}{\mathrm{Lt}} \frac{d^2}{ds^2} F(s) = \underset{s\to 0}{\mathrm{Lt}} \int_0^t T^2\, f(T)\, e^{-st}\, dT$$

$$= \int_0^t T^2\, f(T) \underset{s\to 0}{\mathrm{Lt}}\, e^{-st}\, dT = \int_0^t T^2\, f(T)\, dT = C_2$$

$$\therefore \boxed{C_2 = \underset{s\to 0}{\mathrm{Lt}} \frac{d^2}{ds^2} F(s)} \qquad .....(2.72)$$

Similarly it can be shown that,

$$\boxed{C_n = \underset{s\to 0}{\mathrm{Lt}} \frac{d^n}{ds^n} F(s)} \qquad .....(2.73)$$

2.17 CORRELATION BETWEEN STATIC AND DYNAMIC ERROR COEFFICIENTS

The values of dynamic error coefficients can be used to calculate static error coefficients. The following expressions shows the relationship between them.

$$C_0 = \frac{1}{1+K_p} \qquad .....(2.74)$$

$$C_1 = \frac{1}{K_v} \qquad .....(2.75)$$

$$C_2 = \frac{1}{K_a} \qquad .....(2.76)$$

Proof

$$C_0 = \underset{s\to 0}{\mathrm{Lt}}\, F(s) = \underset{s\to 0}{\mathrm{Lt}} \frac{1}{1+G(s)\,H(s)} = \frac{1`}{1+\underset{s\to 0}{\mathrm{Lt}}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

2.18 ALTERNATE METHOD FOR GENERALIZED ERROR COEFFICIENTS

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)\,H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} \qquad .....(2.77)$$

The equation (2.77) can be expressed as a power series of s as shown in equation (2.78).

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + ... \qquad .....(2.78)$$

$$\therefore E(s) = C_0\, R(s) + C_1 s R(s) + \frac{C_2}{2!}s^2\, R(s) + \frac{C_3}{3!}s^3\, R(s) + ... \qquad .....(2.79)$$

On taking inverse Laplace transform of equation (2.79) we get,

$$e(t) = C_0\, r(t) + C_1 s r(t) + \frac{C_2}{2!}s^2\, r(t) + \frac{C_3}{3!}s^3\, r(t) + ... \qquad .....(2.80)$$

The equation (2.80) is same as that of equation (2.65) in section 2.14.This method will be useful to find the generalized error coefficients without using differentiation, but using laplace transform.

EXAMPLE 2.11

For a unity feedback control system the open loop transfer function, $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$

SOLUTION

a) To find static error constants

For a unity feedback system , H(s)=1

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt}\, \dfrac{10(s+2)}{s^2(s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, s\, G(s)H(s) = \underset{s\to 0}{Lt}\, s\, G(s) = \underset{s\to 0}{Lt}\, s\dfrac{10(s+2)}{s^2(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt}\, s^2G(s)$

$$= \underset{s\to 0}{Lt}\, s^2\, \frac{10(s+2)}{s^2(s+1)} = \frac{10\times 2}{1} = 20$$

b) To find steady state error

Method-I

Steady state error for non-standard input is obtained using generalized error series, given below.

The error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\dfrac{C_2}{2!} + \ldots\ldots + \overset{n}{r}(t)\dfrac{C_n}{n!} + \ldots.$

Given that, $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$

Input signal in time domain, $r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}\right\}$

$$= 3 - 2t + \frac{1}{3}\frac{t^2}{2!} = 3 - 2t + \frac{t^2}{6}$$

$$\therefore \dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

The derivatives of r(t) is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2. The generalized error constants are given by,

$$C_0 = \underset{s\to 0}{Lt}\, F(s)\,; \qquad C_1 = \underset{s\to 0}{Lt}\, \frac{d}{ds}F(s)\,; \qquad C_2 = \underset{s\to 0}{Lt}\, \frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)} = \frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0 = \underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right] = 0$$

$$C_1 = \underset{s\to 0}{Lt}\frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)(3s^2+2s)-(s^3+s^2)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{3s^5+2s^4+3s^4+2s^3+30s^3+20s^2+60s^2+40s-3s^5-2s^4-10s^3-3s^4-2s^3-10s^2}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0$$

$$C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40) - (20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)\,(3s^2+2s+10)}{(s^3+s^2+10s+20)^4}\right] = \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\dfrac{C_2}{2!} = \left(3-2t+\dfrac{t^2}{6}\right)\times 0 + \left(-2+\dfrac{t}{3}\right)\times 0 + \dfrac{1}{3}\times\dfrac{1}{10}\times\dfrac{1}{2!} = \dfrac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\,\dfrac{1}{60} = \dfrac{1}{60}$

Method - II

The error signal in s - domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

Given that, $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$; $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore E(s) = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

Method - III

Error signal in s - domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

Given that, $G(s) = \frac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20} = \frac{s^2+s^3}{20+10s+s^2+s^3} = \frac{s^2}{20}+\frac{s^3}{40}+....$$

$$E(s) = R(s)\left[\frac{s^2}{20}+\frac{s^3}{40}+...\right] = \frac{1}{20}s^2R(s)+\frac{1}{40}s^3R(s)+...$$

$$\begin{array}{r|l} & \frac{s^2}{20}+\frac{s^3}{40}\cdots\cdots \\ \hline 20+10s+s^2+s^3 & s^2+s^3 \\ & \underset{(-)}{s^2}\underset{(-)}{+}\frac{s^3}{2}\underset{(-)}{+}\frac{s^4}{20}\underset{(-)}{+}\frac{s^5}{20} \\ \hline & \frac{s^3}{2}-\frac{s^4}{20}-\frac{s^5}{20} \\ & \underset{(-)}{\frac{s^3}{2}}\underset{(-)}{+}\frac{s^4}{4}\underset{(-)}{+}\frac{s^5}{40}\underset{(-)}{+}\frac{s^6}{40} \\ \hline & -\frac{3s^4}{10}-\frac{3s^5}{40}-\frac{s^6}{40} \\ & \vdots \end{array}$$

Dividing numerator polynomial by denominator polynomial.

On taking inverse Laplace transform of the above equation we get,

$$e(t) = \frac{1}{20}\ddot{r}(t)+\frac{1}{40}\dddot{r}(t)+...$$

Given that, $R(s) = \frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}$

$$\therefore r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}\right\} = 3-2t+\frac{1}{3}\frac{t^2}{2!} = 3-2t+\frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2+\frac{1}{6}2t = -2+\frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain, $e(t) = \frac{1}{20}\ddot{r}(t) = \frac{1}{20}\left(\frac{1}{3}\right) = \frac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\,\frac{1}{60} = \frac{1}{60}$

RESULT

(a) Position error constant, $K_p = \infty$

Velocity error constant, $K_v = \infty$

Acceleration error constant, $K_a = 20$

(b) When, $R(s) = \frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}$, Steady state error, $e_{ss} = \frac{1}{60}$

EXAMPLE 2.12

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \frac{10}{(s+2)(s+3)}$; c) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

SOLUTION

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\ \frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

b) $G(s) = \dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \dfrac{10}{(s+2)(s+3)} = \dfrac{10}{2\times 3} = \dfrac{5}{3}$

Steady state error, $e_{ss} = \dfrac{1}{1+K_p} = \dfrac{1}{1+\frac{5}{3}} = \dfrac{3}{3+5} = \dfrac{3}{8} = 0.375$

c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \dfrac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2\,G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\,G(s) = \underset{s\to 0}{Lt}\ s^2\ \dfrac{10}{s^2(s+1)(s+2)} = \dfrac{10}{1\times 2} = 5$

Steady state error, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{5} = 0.2$

RESULT

1. In system (a) with unit velocity input, Steady state error = 0.075
2. In system (b) with unit step input, Steady state error = 0.375
3. In system (c) with unit acceleration input, Steady state error = 0.2

EXAMPLE 2.13

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynominal, $r(t) = a_0 + a_1 t + \frac{a_2}{2}t^2$.

SOLUTION

To find static error constant

For unity feedback system, H(s) = 1.

$\therefore$ Loop transfer function, G(s) H(s) = G(s)

The static error constants are K_p, K_v and K_a.

$$\text{Position error constant, } K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{s(0.1s+1)} = \infty$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\frac{10}{s(0.1s+1)} = 10$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ s^2\frac{10}{s(0.1s+1)} = 0$$

To find steady state error

Method - I

Steady state error for non-standard input is obtained using generalized error series, given below.

$$\text{The error signal, } e(t) = r(t)C_o + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + ... + \overset{n}{r}(t)\frac{C_n}{n!} + ...$$

$$\text{Given that, } r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$$

$$\therefore\ \dot{r}(t) = \frac{d}{dt}r(t) = \frac{d}{dt}\left(a_o + a_1 t + \frac{a_2}{2}t^2\right) = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\left(\frac{d}{dt}r(t)\right) = \frac{d}{dt}(a_1 + a_2 t) = a_2$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\left(\frac{d^2}{dt^2}r(t)\right) = \frac{d}{dt}(a_2) = 0$$

Derivatives of r(t) is zero after 2nd derivative. Hence, let us evaluate three constants C_0, C_1 & C_2.

The generalized error constants are given by,

$$C_o = \underset{s\to 0}{Lt}\ F(s)\ ; \qquad C_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds}F(s)\ ; \qquad C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1\,s^2+s}{0.1s^2+s+10}$$

$$C_o = \underset{s\to 0}{Lt}\ F(s) = \underset{s\to 0}{Lt}\ \frac{0.1s^2+s}{0.1s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(0.1s^2+s+10)\,(0.2s+1) - (0.1s^2+s)\,(0.2s+1)}{(0.1s^2+s+10)^2}\right] = \underset{s\to 0}{Lt}\ \frac{2s+10}{(0.1s^2+s+10)^2} = \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{(0.1s^2+s+10)^2 \times 2 - (2s+10)\times 2\,(0.1s^2+s+10)\,(0.2s+1)}{(0.1s^2+s+10)^4}\right]$$

$$\therefore C_2 = \frac{10^2\times 2 - 10\times 2\times 10\times 1}{10^4} = 0$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \dot{r}(t)C_1 + 0 + 0 = (a_1 + a_2 t)\ 0.1$

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\ \left[(a_1 + a_2 t)\ 0.1\right] = \infty$

Method - II

The error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$; $\quad G(s) = \frac{10}{s(0.1s+1)}$; $\quad H(s) = 1$

On taking Laplace transform of r(t) we get R(s),

$$\therefore R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1 + \frac{10}{s(0.1s+1)}} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{\frac{s(0.1s+1)+10}{s(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_o s(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s[s(0.1s+1)+10]}\right\} = 0 + \frac{a_1}{10} + \infty = \infty$$

Method - III

Error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$; $\qquad \therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$

Given that, $G(s) = \frac{10}{s(0.1s+1)}$ and $H(s) = 1$.

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10} = \frac{s+0.1s^2}{10+s+0.1s^2} = \frac{s}{10} - \frac{s^3}{1000} + \ldots$$

Dividing numerator polynomial by denominator polynomial.

$$\therefore E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + \ldots$$

On taking inverse Laplace transform,

$$e(t)=\frac{1}{10}\dot{r}-\frac{1}{1000}\dddot{r}(t)+.....$$

$$
\begin{array}{r|l}
 & \dfrac{s}{10}-\dfrac{s^3}{1000} \\
\hline
10+s+0.1s^2 & s+0.1s^2 \\
 & \underset{(-)}{s}+\underset{(-)}{\dfrac{s^2}{10}}+\underset{(-)}{\dfrac{s^3}{100}} \\
\hline
 & -\dfrac{s^3}{100} \\
 & \underset{(+)}{-\dfrac{s^3}{100}}\underset{(+)}{-\dfrac{s^4}{1000}}\underset{(+)}{-\dfrac{s^5}{10000}} \\
\hline
 & \dfrac{s^4}{1000}+\dfrac{s^5}{10000}
\end{array}
$$

Given that, $r(t)=a_0+a_1t+\frac{a_2}{2}t^2$

$$\therefore\ \dot{r}=\frac{d}{dt}r(t)=a_1+a_2t$$

$$\ddot{r}(t)=\frac{d}{dt}\dot{r}(t)=a_2$$

$$\dddot{r}(t)=\frac{d}{dt}\ddot{r}(t)=0$$

$\therefore$ Error signal in time domain, $e(t)=\frac{1}{10}\dot{r}(t)=\frac{1}{10}(a_1+a_2t)$

Steady state error, $e_{ss}=\underset{t\to\infty}{Lt}\ e(t)=\underset{t\to\infty}{Lt}\ \frac{1}{10}(a_1+a_2t)=\infty$

RESULT

(a) Position error constant, $K_p=\infty$

(b) Velocity error constant, $K_v=10$

(c) Acceleration error constant, $K_a=0$

(d) When input, $r(t)=a_o+a_1t+\frac{a_2t^2}{2}$, Steady state error, $e_{ss}=\infty$

EXAMPLE 2.14

Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)}=\frac{Ks+b}{s^2+as+b}$. Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by $\frac{(a-K)}{b}$.

SOLUTION

For unity feedback system, H(s)=1

The closed loop transfer function, $M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)}$

$$\therefore\frac{G(s)}{1+G(s)}=M(s)$$

On cross multiplication of the above equation we get,

$$G(s)=M(s)[1+G(s)]=M(s)+M(s)\,G(s)$$

$$\therefore\ G(s)-M(s)\,G(s)=M(s)\quad\Rightarrow\quad G(s)[1-M(s)]=M(s)\quad\Rightarrow\quad M(s)=\frac{Ks+b}{s^2+as+b}$$

$\therefore$ Open loop transfer function,

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{\left(s^2+as+b\right)-(Ks+b)}$$

$$=\frac{Ks+b}{s^2+as+b-Ks-b}=\frac{Ks+b}{s^2+(a-k)s}=\frac{Ks+b}{s[s+(a.K)]}$$

Velocity error constant, $K_V = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\frac{Ks+b}{s[s+(a-K)]} = \frac{b}{a-K}$

With velocity input, Steady state error, $e_{ss} = \frac{1}{K_V} = \frac{a-K}{b}$

RESULT

Open loop transfer function, $G(s) = \frac{Ks+b}{s[s+(a-K)]}$

With velocity input, Steady state error, $e_{ss} = \frac{a-K}{b}$

EXAMPLE 2.15

A unity feedback system has the forward transfer function $G(s) = \frac{K_1(2s+1)}{s(5s+1)(1+s)^2}$. When the input $r(t) = 1+6t$, determine the minimum value of K_1 so that the steady error is less than 0.1.

SOLUTION

Given that, input $r(t) = 1+6t$

On taking laplace transform of r(t) we get R(s).

$$\therefore R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1+6t\} = \frac{1}{s}+\frac{6}{s^2}$$

The error signal in s-domain E(s) is given by,

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{K_1(2s+1)}{s(5s+1)(1+s)^2}} = \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s(5s+1)(1+s)^2+K_1(2s+1)}{s(5s+1)(1+s)^2}}$$

Here H(s) = 1

$$= \frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$$

$$= \underset{s\to 0}{Lt}\ s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)} + \frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\} = 0+\frac{6}{K_1} = \frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$, $\quad \therefore 0.1 = \frac{6}{K_1} \quad$ or $\quad K_1 = \frac{6}{0.1} = 60$

RESULT

For steady state error, $e_{ss} < 0.1$, the value of K_1 should be greater than 60.

2.19 COMPONENTS OF AUTOMATIC CONTROL SYSTEM

The basic components of an automatic control system are Error detector, Amplifier and Controller, Actuator (Power actuator), Plant and Sensor or Feedback system. The block diagram of an automatic control system is shown in fig 2.16.

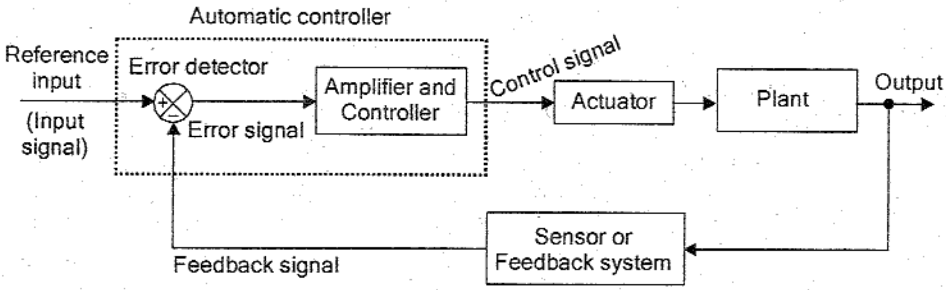


Fig 2.16: Block diagram of automatic control system.

The plant is the open loop system whose output is automatically controlled by closed loop system. The combined unit of error detector, amplifier and controller is called ***automatic controller***, because without this unit the system becomes open loop system.

In automatic control systems the reference input will be an input signal proportional to desired output. The feedback signal is a signal proportional to current output of the system. The error detector compares the reference input and feedback signal and if there is a difference it produces an error signal. An amplifier can be used to amplify the error signal and the controller modifies the error signal for better control action.

The actuator amplifies the controller output and converts to the required form of energy that is acceptable for the plant. Depending on the input to the plant, the output will change. This process continues as long as there is a difference between reference input and feedback signal. If the difference is zero, then there is no error signal and the output settles at the desired value.

Generally, the error signal will be a weak signal and so it has to be amplified and then modified for better control action. In most of the system the controller itself amplifies the error signal and integrates or differentiates to produce a control signal (i.e., modified error signal). The different types of controllers are P, PI, PD and PID controllers.

2.20 CONTROLLERS

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The manner in which the controller produces the control signal is called the ***control action***. The controller modifies the transient response of the system. The electronic controllers using operational amplifiers are presented in this section.

The following six basic control actions are very common among industrial analog controllers.

1. Two-position or ON-OFF control action.
2. Proportional control action.
3. Integral control action.
4. Proportional- plus- integral control action.
5. Proportional-plus-derivative control action.
6. Proportional-plus-integral-plus-derivative control action.

Depending on the control actions provided the controllers can be classified as follows.

1. Two position or ON-OFF controllers.
2. Proportional controllers.
3. Integral controllers.
4. Proportional-plus-integral controllers.
5. Proportional-plus-derivative controllers.
6. Proportional-plus-integral-plus-derivative controllers.

ON-OFF (OR) TWO POSITION CONTROLLER

The ON-OFF or two position controller has only two fixed positions. They are either on or off. The on-off control system is very simple in construction and hence less expensive. For this reason, it is very widely used in both industrial and domestic control systems.

The ON-OFF control action may be provided by a relay. There are different types of relay. The most popular one is electromagnetic relay. It is a device which has NO (Normally Open) and NC (Normally Closed) contacts, whose opening and closing are controlled by the relay coil. When the relay coil is excited, the relay operates and the contacts change their positions (i.e., NO $\rightarrow$ NC and NC $\rightarrow$ NO).

Let the output signal from the controller be u(t) and the actuating error signal be e(t). In this controller, u(t) remains at either a maximum or minimum value.

$$u(t) = u_1 \,; \quad \text{for } e(t) < 0$$
$$\quad\;\; = u_2 \,; \quad \text{for } e(t) > 0$$

$$\boxed{E(s) = \mathcal{L}\{e(t)\} \,; \quad U(s) = \mathcal{L}\{u(t)\}}$$

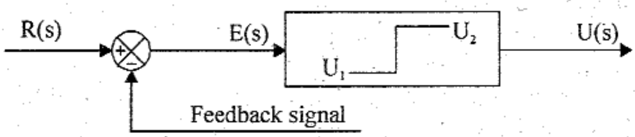


Fig 2.17 : Block diagram of on-off controller.

PROPORTIONAL CONTROLLER (P - CONTROLLER)

The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal, e(t).

In P-controller, $u(t) \propto e(t)$

$$\therefore \quad u(t) = K_p\, e(t) \qquad .....(2.81)$$

where, K_p = Proportional gain or constant

On taking Laplace transform of equation (2.81) we get,

$$U(s) = K_p E(s) \qquad .....(2.82)$$

$$\boxed{\therefore \text{ Transfer function of P-controller, } \frac{U(s)}{E(s)} = K_p} \qquad .....(2.83)$$

The equation (2.82) gives the output of the P-controller for the input E(s) and equation (2.83) is the transfer function of the P-controller. The block diagram of the P-controller is shown in fig 2.18.

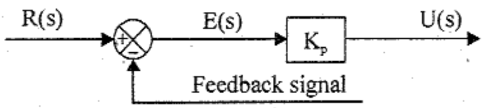


Fig 2.18 : Block diagram of proportional controller.

From the equation (2.82), we can conclude that the proportional controller amplifies the error signal by an amount K_p. Also the introduction of the controller on the system increases the loop gain by an amount K_p. The increase in loop gain improves the steady state tracking accuracy, disturbance signal rejection and the relative stability and also makes the system less sensitive to parameter variations. But increasing the gain to very large values may lead to instability of the system. The drawback in P-controller is that it leads to a constant steady state error.

EXAMPLE OF ELECTRONIC P-CONTROLLER

The proportional controller can be realized by an amplifier with adjustable gain. Either the non-inverting operational amplifier or the inverting operational amplifier followed by sign changer will work as a proportional controller. The op-amp proportional controller is shown in fig 2.19 and 2.20.

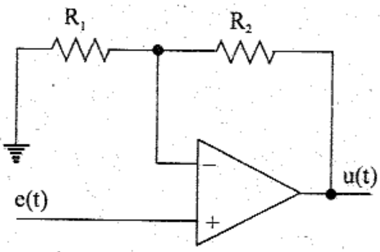


Fig 2.19 : Op-amp P-controller using non-inverting amplifier.

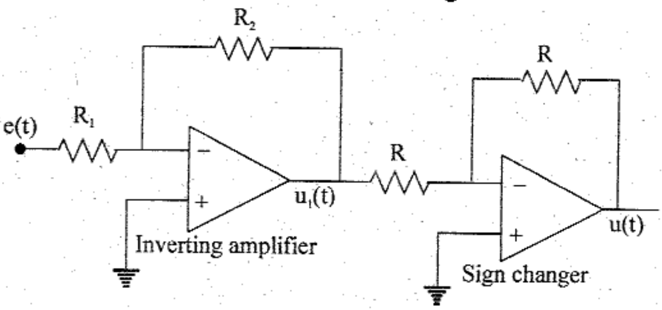


Fig 2.20 : Op-amp P-controller using inverting amplifier.

By deriving the transfer function of the controllers shown in fig 2.11 and 2.12 and comparing with the transfer function of P-controller defined by equation (2.83), it can be shown that they work as P-controllers.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.19

In fig 2.19, the input e(t) is applied to positive input. By symmetry of op-amp the voltage of negative input is also e(t). Also we assume an ideal op-amp so that input current is zero. Based on the above assumptions the equivalent circuit of the controller is shown in fig 2.21.

By voltage division rule,

$$e(t) = \frac{R_1}{R_1 + R_2}\, u(t) \ ; \qquad \therefore u(t) = \frac{R_1 + R_2}{R_1}\, e(t) \qquad .....(2.84)$$

On taking Laplace transform of equation (2.84) we get,

$$U(s) = \frac{R_1 + R_2}{R_1}\, E(s) \qquad .....(2.85)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_1 + R_2}{R_1} \qquad .....(2.86)$$

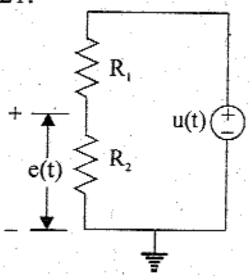


Fig 2.21 : Equivalent circuit of P-controller shown in fig 2.19.

The equation (2.86) is the transfer function of op-amp P-controller. On comparing equation (2.86) with equation (2.83) we get,

$$\text{Proportional gain, } K_P = \frac{R_1 + R_2}{R_1} \qquad .....(2.87)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain, K_p can be varied.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.20

The assumption made in op-amp circuit analysis are,

1. The voltages at both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.22 and 2.23.

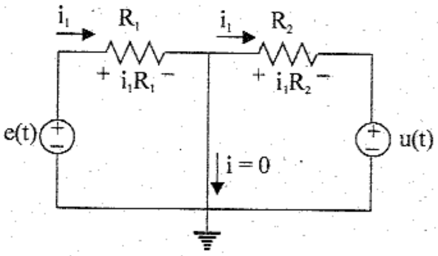


Fig 2.22 : Equivalent circuit of amplifier.

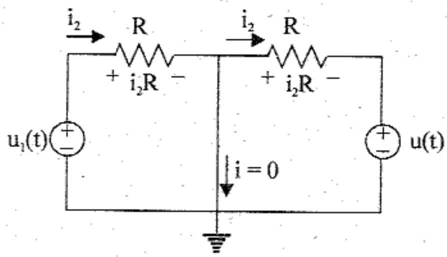


Fig 2.23 : Equivalent circuit of sign changer.

From fig 2.22, $e(t) = i_1R_1$; $\therefore\ i_1 = \dfrac{e(t)}{R_1}$ .....(2.88)

$u_1(t) = -i_1R_2$ .....(2.89)

Substitute for i_1 from equation (2.88) in equation (2.89).

$$\therefore\ u_1(t) = -\frac{e(t)}{R_1}R_2 \qquad .....(2.90)$$

From fig 2.23, $u(t) = -i_2R$; $\therefore i_2 = -\dfrac{u(t)}{R}$ .....(2.91)

$u_1(t) = i_2R$ .....(2.92)

Substitute for i_2 from equation (2.91) in equation (2.92).

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.93)$$

On equating the equations (2.90) and (2.93) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2\ ; \qquad u(t) = \frac{R_2}{R_1}e(t) \qquad .....(2.94)$$

On taking Laplace transform of equation (2.94) we get,

$$U(s) = \frac{R_2}{R_1}E(s) \qquad .....(2.95)$$

$$\therefore\quad \frac{U(s)}{E(s)} = \frac{R_2}{R_1} \qquad .....(2.96)$$

The equation (2.96) is the transfer function of op-amp P-controller. On comparing equation (2.96) with equation (2.83) we get,

$$\text{Proportional gain, } K_p = \frac{R_2}{R_1} \qquad .....(2.97)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain K_p can be varied.

INTEGRAL CONTROLLER (I-CONTROLLER)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal, e(t).

In I-controller, $u(t) \; \alpha \int e(t)\,dt$; $\therefore u(t) = K_i \int e(t)\,dt$ .....(2.98)

where, K_i = Integral gain or constant.

On taking Laplace transform of equation (2.98) with zero initial conditions we get,

$$U(s) = K_i \frac{E(s)}{s} \quad .....(2.99)$$

$$\boxed{\therefore \text{ Transfer function of I-controller, } \frac{U(s)}{E(s)} = \frac{K_i}{s}} \quad .....(2.100)$$

The equation (2.99) gives the output of the I-controller for the input E(s) and equation (2.101) is the transfer function of the I-controller. The block diagram of I-controller is shown in fig 2.24.

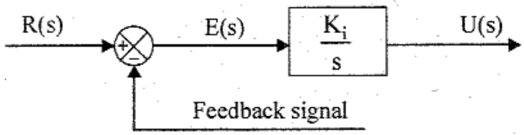


Fig 2.24 : Block diagram of an integral controller.

The integral controller removes or reduces the steady error without the need for manual reset. Hence the I-controller is sometimes called ***automatic reset***. The drawback in integral controller is that it may lead to oscillatory response of increasing or decreasing amplitude which is undesirable and the system may become unstable.

EXAMPLE OF ELECTRONIC I-CONTROLLER

The integral controller can be realized by an integrator using op-amp followed by a sign changer as shown in fig 2.25.

By deriving the transfer function of the controller shown in fig 2.25 and comparing with the transfer function of I-controller defined by equation(2.101), it can be shown that it work as I-controller.

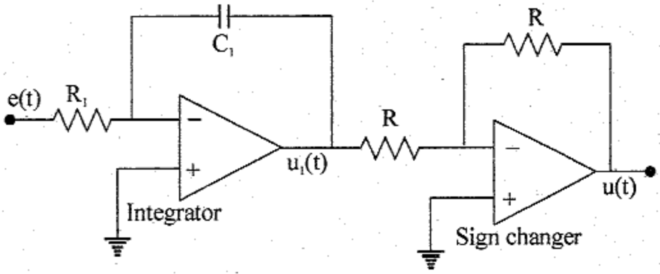


Fig 2.25 : I-controller using op-amp.

ANALYSIS OF I-CONTROLLER SHOWN IN FIG 2.25

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.26 and 2.27.

From fig 2.26, $e(t) = i_1 R_1$; $\therefore i_1 = \frac{e(t)}{R_1}$ .....(2.101)

$$u_1(t) = -\frac{1}{C_1}\int i_1 dt \quad .....(2.102)$$

Substitute for i_1 from equation (2.101) in equation (2.102).

$$\therefore\ u_1(t) = -\frac{1}{C_1}\int \frac{e(t)}{R_1}dt = -\frac{1}{R_1C_1}\int e(t)dt \qquad .....(2.103)$$

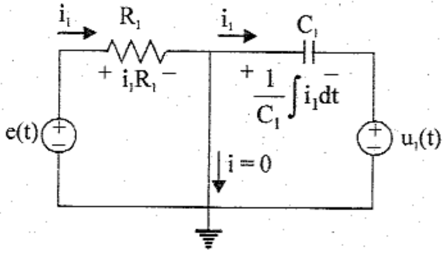


Fig 2.26 : Equivalent circuit of integrator.

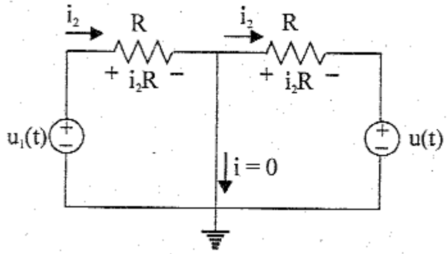


Fig 2.27 : Equivalent circuit of sign changer.

From fig 2.27, $u(t) = -i_2R, \ \therefore\ i_2 = \frac{-u(t)}{R}$ .....(2.104)

$$u_1(t) = i_2R \qquad .....(2.105)$$

Substitute for i_2 from equation (2.106) in equation (2.107),

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.106)$$

On equating the equations (2.103) and (2.106) we get,

$$-u(t) = -\frac{1}{R_1C_1}\int e(t)\,dt$$

$$\therefore\ u(t) = \frac{1}{R_1C_1}\int e(t)\,dt \qquad .....(2.107)$$

On taking Laplace transform of equation (2.107) with zero initial conditions we get,

$$U(s) = \frac{1}{R_1C_1}\frac{E(s)}{s} \qquad .....(2.108)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{1}{R_1C_1}\frac{1}{s} \qquad .....(2.109)$$

The equation (2.109) is the transfer function of op-amp I-controller. On comparing equation (2.109) with equation (2.100) we get,

$$\text{Integral gain, } K_i = \frac{1}{R_1C_1} \qquad .....(2.110)$$

Therefore by adjusting the values of R_1 and C_1 the value of gain K_i can be varied.

PROPORTIONAL PLUS INTEGRAL CONTROLLER (PI-CONTROLLER)

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

In PI - controller, $u(t)\ \alpha\ \left[e(t) + \int e(t)\,dt\right]$; $\quad \therefore u(t) = K_p\,e(t) + \frac{K_p}{T_i}\int e(t)dt$ ....(2.111)

where, K_p = Proportional gain

T_i = Integral time.

On taking Laplace transform of equation (2.111) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + \frac{K_p}{T_i}\,\frac{E(s)}{s} \qquad .....(2.112)$$

$$\boxed{\therefore \text{ Transfer function of PI - controller, } \frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right)} \qquad .....(2.113)$$

The equation (2.112) gives the output of the PI-controller for the input E(s) and equation (2.113) is the transfer function of the PI controller. The block diagram of PI-controller is shown in fig 2.28.

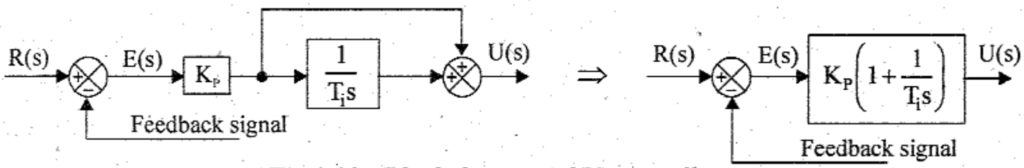


Fig 2.28 : Block diagram of PI-controller.

The advantages of both P-controller and I-controller are combined in PI-controller. The proportional action increases the loop gain and makes the system less sensitive to variations of system parameters. The integral action eliminates or reduces the steady state error.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proportional and integral parts of control action. The inverse of the integral time T_i is called the ***reset rate.***

EXAMPLE OF ELECTRONIC PI-CONTROLLER

The PI-controller can be realized by an op-amp integrator with gain followed by a sign changer as shown in fig 2.29.

By deriving the transfer function of the controller shown in fig (2.29) and comparing with the transfer function of PI-controller defined by equation (2.114), it can be proved that the circuit shown in fig 2.29, work as PI-controller.

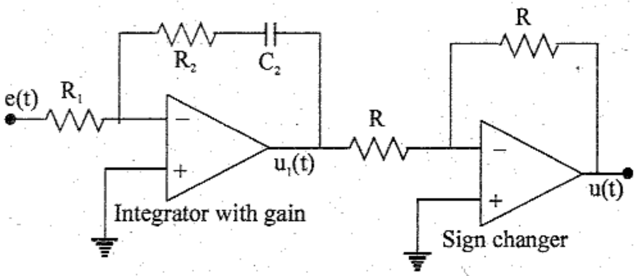


Fig 2.29 : PI-controller using op-amp.

ANALYSIS OF PI-CONTROLLER SHOWN IN FIG 2.29

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.30. and 2.31.

From fig 2.30, $e(t) = i_1 R_1$; $\therefore\ i_1 = \frac{e(t)}{R_1}$ .....(2.114)

$$u_1(t) = -i_1 R_2 - \frac{1}{C_2}\int i_1 dt \qquad .....(2.115)$$

Substitute for i_1 from equation (2.114) in equation (2.115).

$$\therefore\ u_1(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt \qquad .....(2.116)$$

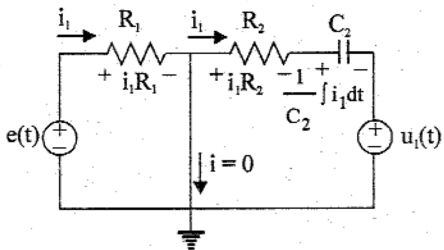


Fig 2.30 : Equivalent circuit of integrator.

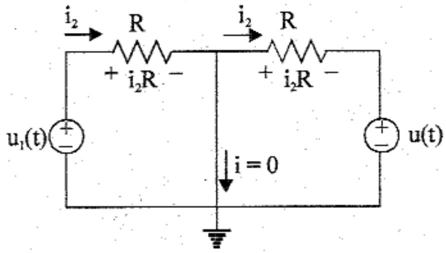


Fig 2.31 : Equivalent circuit of sign changer.

From fig 2.31, $u(t) = -i_2R, \;\; \therefore \; i_2 = \frac{-u(t)}{R}$ .....(2.117)

$u_1(t) = i_2R$ .....(2.118)

Substitute for i_2 from equation (2.117) in equation (2.118),

$$\therefore \; u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.119)$$

On equating the equations (2.116) and (2.119) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt$$

$$\therefore \; u(t) = \frac{R_2}{R_1}e(t) + \frac{1}{R_1C_2}\int e(t)\,dt \qquad .....(2.120)$$

On taking Laplace transform of equation (2.120) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}E(s) + \frac{1}{R_1C_2}\frac{E(s)}{s}$$

$$\therefore \; \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{R_2C_2s}\right) \qquad .....(2.121)$$

The equation (2.121) is the transfer function of op-amp PI-controller. On comparing equation(2.121) with equation (2.113) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$; Integral time, $T_i = R_2C_2$

By varying the values of R_1 and R_2, the value of gain K_p and T_i can be adjusted.

PROPORTIONAL PLUS DERIVATIVE CONTROLLER (PD-CONTROLLER)

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

In PD - controller, $u(t) \; \alpha \; \left[e(t) + \frac{d}{dt}e(t)\right]$; $\therefore u(t) = K_p\,e(t) + K_pT_d\frac{d}{dt}e(t)$ .....(2.122)

where, K_p = Proportional gain

T_d = Derivative time

On taking Laplace transform of equation (2.123) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + K_p\,T_d\,sE(s) \qquad .....(2.123)$$

$$\therefore \text{ Transfer function of PD-controller, } \frac{U(s)}{E(s)} = K_p(1+T_d s) \qquad .....(2.124)$$

The equation (2.123) gives the output of the PD-controller for the input E(s) and equation (2.124) is the transfer function of PD-controller.

The block diagram of PD-controller is shown in fig 2.32.

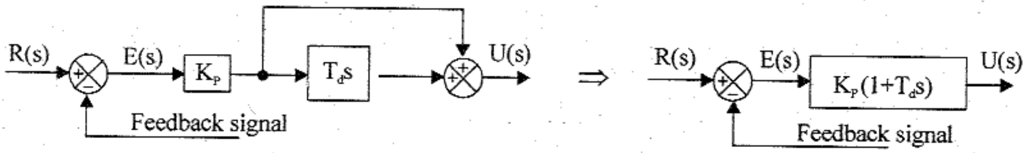


Fig 2.32 : Block diagram of PD- controller.

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control action is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers. The derivative controller does not affect the steady-state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system. While derivative control action has an advantage of being anticipatory it has the disadvantage that it amplifies noise signals and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both the proportional and derivative parts of control action. The derivative control is also called ***rate control.***

EXAMPLE OF ELECTRONIC PD-CONTROLLER

The PD-controller can be realized by an op-amp differentiator with gain followed by a sign changer as shown in fig 2.33.

By deriving the transfer function of the controller shown in fig 2.33 and comparing with the transfer function of PD-controller defined by equation (2.124) it can be proved that the circuit shown in fig 2.33 will work as PD-controller.

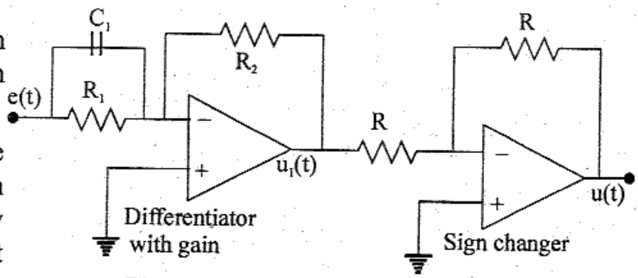


Fig 2.33 : PD controller using op-amp.

ANALYSIS OF PD-CONTROLLER SHOWN IN FIG 2.33

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp differentiator and sign changer are shown in fig 2.34 and 2.35.

From fig 2.34, $\therefore\ i_1 = \dfrac{e(t)}{R_1} + C_1 \dfrac{de(t)}{dt}$ .....(2.125)

$$i_1 R_2 = -u_1(t), \qquad \therefore\ i_1 = \frac{-u_1(t)}{R_2} \qquad .....(2.126)$$

On equating the equations (2.125) and (2.126) we get,

$$-\frac{u_1(t)}{R_2} = \frac{e(t)}{R_1} + C_1 \frac{d}{dt} e(t)\,; \qquad \therefore\ u_1(t) = -\left(\frac{R_2}{R_1} e(t) + R_2 C_1 \frac{d}{dt} e(t)\right) \qquad .....(2.127)$$

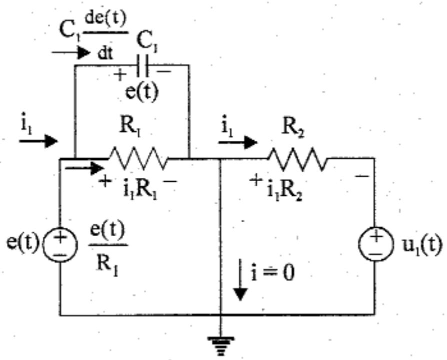


Fig 2.34 : Equivalent circuit of differentiator.

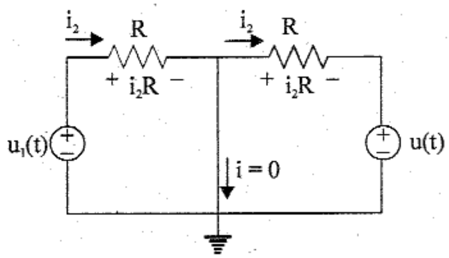


Fig 2.35 : Equivalent circuit of sign changer.

From fig 2.35, $u(t) = -i_2R$; $\therefore\ i_2 = \frac{-u(t)}{R}$ .....(2.128)

$$u_1(t) = i_2R \quad .....(2.129)$$

Substitute for i_2 from equation (2.128) in equation (2.129).

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \quad .....(2.130)$$

On equating the equations (2.127) and (2.130) we get,

$$-u(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right)$$

$$\therefore\ u(t) = \frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t) \quad .....(2.131)$$

On taking Laplace transform of equation (2.131) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}\ E(s) + R_2C_1\ s\ E(s) \quad .....(2.132)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\ (1 + R_1C_1\ s) \quad .....(2.133)$$

The equation (2.133) is the transfer function of op-amp PD-controller. On comparing equation (2.133) with equation (2.124) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$

Derivative time, $T_d = R_1C_1$

By varying the values of R_1 and R_2, the value of gain K_p and T_d can be adjusted.

PROPORTIONAL PLUS INTEGRAL PLUS DERIVATIVE CONTROLLER (PID-CONTROLLER)

The PID-controller produces an output signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

In PID - controller, $u(t) \; \alpha \; \left[e(t) + \int e(t)\, dt + \frac{d}{dt} e(t)\right]$

$$\therefore u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt} e(t) \qquad .....(2.134)$$

where, K_p = Proportional gain

T_i = Integral time

T_d = Derivative time

On taking Laplace transform of equation (2.134) with zero initial conditions we get,

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\;\frac{E(s)}{s} + K_p\, T_d\, s\, E(s) \qquad .....(2.135)$$

$$\boxed{\therefore \text{ Transfer function of PID - controller, } \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)} \qquad .....(2.136)$$

The equation (2.135) gives the output of the PID-controller for the input E(s) and equation (2.136) is the transfer function of the PID-controller. The block diagram of PID-contoller is shown in fig 2.36.

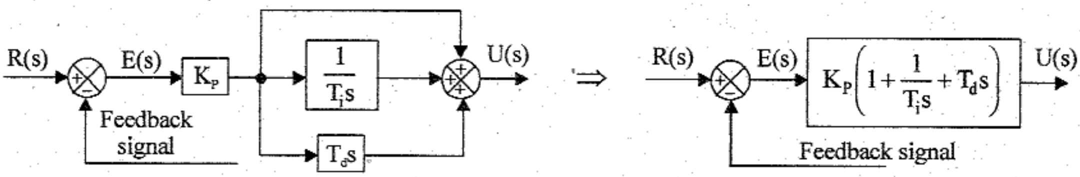


Fig 2.36: Block diagram of PID- controller.

The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

EXAMPLE OF ELECTRONIC PID-CONTROLLER

The PID-controller can be realized by op-amp amplifier with integral and derivative action followed by sign changer as shown in fig 2.37.

By deriving the transfer function of the controller shown in fig (2.37) and comparing with the transfer function of PID-controller defined by equation (2.136) it can be proved that the circuit shown in fig 2.37 work as PID-controller.

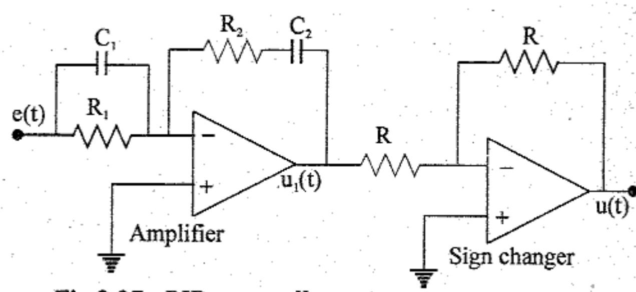


Fig 2.37 : PID- controller using op-amp.

ANALYSIS OF PID-CONTROLLER SHOWN IN FIG 2.37

The assumptions made in op-amp circuit analysis are.

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.38 and 2.39.

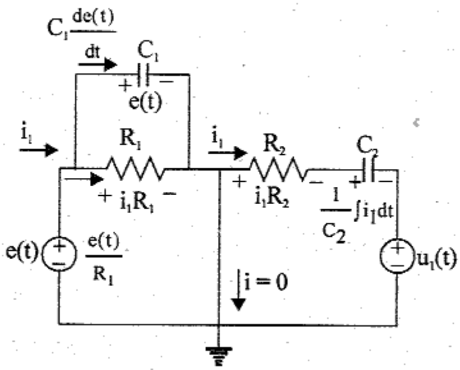


Fig 2.38 : Equivalent circuit of amplifier.

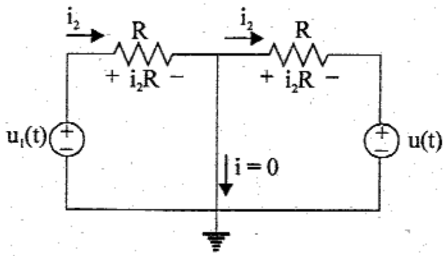


Fig 2.39 : Equivalent circuit of sign changer.

From fig 2.38, $i_1 = \dfrac{e(t)}{R_1} + C_1 \dfrac{de(t)}{dt}$ .....(2.137)

On taking Laplace transform of equation (2.137) with zero initial conditions we get,

$$I_1(s) = \frac{1}{R_1}\,E(s) + C_1\,s\,E(s)$$

$$I_1(s) = \left(\frac{1}{R_1} + C_1 s\right) E(s) \quad .....(2.138)$$

From fig 2.38, $i_1 R_2 + \dfrac{1}{C_2}\displaystyle\int i_1 dt = -u_1(t)$ .....(2.139)

On taking Laplace transform of equation (2.138) with zero initial conditions we get,

$$I_1(s)\,R_2 + \frac{1}{C_2}\,\frac{I_1(s)}{s} = -U_1(s)$$

$$\therefore I_1(s)\left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s) \quad .....(2.140)$$

Substitute for $I_1(s)$ from equation (2.138) in equation (2.140).

$$\therefore \left(\frac{1}{R_1} + C_1 s\right) E(s) \left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s)$$

$$-\left(\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s\right) E(s) = U_1(s) \quad .....(2.141)$$

From fig 2.39, $u(t) = -i_2 R$; $\therefore\ i_2 = -\dfrac{u(t)}{R}$ .....(2.142)

$u_1(t) = i_2 R$ .....(2.143)

Substitute for i_2 from equation (2.142) in equation (2.143).

$$\therefore\ u_1(t) = -\frac{u(t)}{R} R = -u(t) \quad .....(2.144)$$

On taking Laplace transform of equation (2.144) we get,

$$U_1(s) = -U(s) \quad .....(2.145)$$

From equations (2.142) and (2.146) we get,

$$U(s)=\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}+\frac{1}{R_1C_2s}+R_2C_1s\right)E(s)$$

$$\therefore \frac{U(s)}{E(s)}=\left(\frac{R_2C_2+R_1C_1}{R_1C_2}+\frac{1}{R_1C_2s}+R_2C_1s\right)$$

$$=\frac{R_2}{R_1}\left(\frac{R_2C_2+R_1C_1}{R_2C_2}+\frac{1}{R_2C_2s}+R_1C_1s\right) \qquad .....(2.146)$$

The equation (2.146) is the transfer function of op-amp PID-controller. On comparing equation (2.146) with equation (2.136) we get,

Proportional gain, $K_p=\frac{R_2}{R_1}$

Derivative time, $T_d=R_1C_1$; Integral time, $T_i=R_2C_2$

Also, $\frac{R_1C_1+R_2C_2}{R_2C_2}=1$

By varying the values of R_1 and R_2 the values of K_p, T_d and T_i are adjusted.

2.21 RESPONSE WITH P, PI, PD AND PID CONTROLLERS

In feedback control systems a controller may be introduced to modify the error signal and to achieve better control action. The introduction of controllers will modify the transient response and the steady state error of the system. The effects due to introduction of P, PI, PD and PID controllers are discussed in this section.

EFFECT OF PROPORTIONAL CONTROLLER (P-CONTROLLER)

The proportional controller produces an output signal which is proportional to error signal. The transfer function of proportional controller is given below. (Refer equation 2.83).

Transfer function of P-controller, $\frac{U(s)}{E(s)} = K_p$

The term K_p in the transfer function of proportional controller is called the gain of the controller. Hence the proportional controller amplifies the error signal and increases the loop gain of the system. The following aspects of system behaviour are improved by increasing loop gain.

* Steady state tracking accuracy.
* Disturbance signal rejection.
* Relative stability.

In addition to increase in loop gain it decreases the sensitivity of the system to parameter variations. The drawback in proportional control action is that it produces a constant steady state error.

EFFECT OF PI-CONTROLLER

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Transfer function of PI-controller, $G_c(s) = K_p\left(1+\frac{1}{T_is}\right)=K_p\left(\frac{T_is+1}{T_is}\right)$ (Refer equation 2.113)

where, K_p is proportional gain and, T_i is integral time.

The block diagram of unity feedback system with PI-controller is shown in fig 2.40.

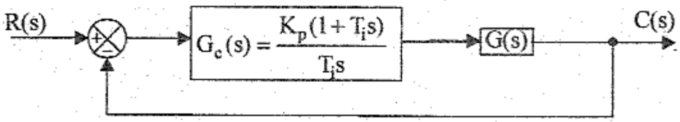


Fig 2.40 : Block diagram of feedback system with PI-controller.

Let the open loop transfer function G(s) be a second order system with transfer function, as shown in equation (2.148).

Open loop transfer function, $G(s) = \dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$ ......(2.147)

Now, loop transfer function $= G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s)$ $\boxed{H(s)=1}$

$$= K_p\left(\frac{1+T_is}{T_is}\right) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_is)}{s^2T_i(s+2\zeta\,\omega_n)} \quad .....(2.148)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_is)}{s^2T_i\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2(1+T_is)}{s^2T_i\,(s+2\zeta\omega_n)}} = \frac{K_p\omega_n^2\,(1+T_is)}{s^2T_i(s+2\zeta\,\omega_n)+K_p\omega_n^2\,(1+T_is)}$$

$$= \frac{K_p\omega_n^2.(1+T_is)}{T_is^3+2\zeta\omega_nT_is^2+K_p\omega_n^2\,T_i\,s+K_p\omega_n^2}$$

$$= \frac{(K_p\,/\,T_i)\,\omega_n^2(1+T_is)}{s^3+2\zeta\omega_ns^2+K_p\omega_n^2\,s+\dfrac{K_p}{T_i}\omega_n^2} \qquad \boxed{K_i = \frac{K_p}{T_i}}$$

$$= \frac{K_i\omega_n^2\,(1+T_is)}{s^3+2\zeta\omega_ns^2+K_p\omega_n^2s+K_i\omega_n^2} \quad .....(2.149)$$

From the closed loop transfer function (equation (3.149)) it is observed that the PI-controller introduces a zero in the system and increases the order by one. The increase in the order of the system results in a less stable system than the original one because higher order systems are less stable than lower order systems.

From the loop transfer function (equation (3.148)) it is observed that the PI-controller increase the type number by one. The increase in type number results in reducing the steady state error. For example if the steady state error of the original system is constant, then the integral controller will reduce the error to zero.

EFFECT OF PD-CONTROLLER

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

The transfer function of PD - controller, $G_c(s) = K_p\,(1+T_d\,s)$ (Refer equation 2.124)

where K_p is Proportional gain, T_d is Derivative time.

The block diagram of unity feedback system with PD-controller is shown in fig 2.41.

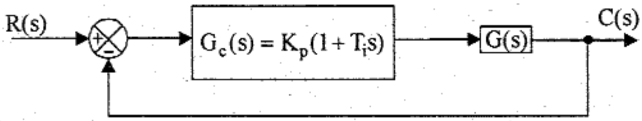


Fig 2.41 : Block diagram of feedback system with PD-controller.

Let the open loop transfer function G(s) be a second order system with transfer function as shown in equation (2.150).

$$\text{Open loop transfer function, } G(s) = \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} \qquad .....(3.150)$$

$$\text{Now, loop transfer function } = G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s) \qquad \boxed{H(s)=1}$$

$$= K_p(1+T_d(s)) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_d s)}{s(s+2\zeta\,\omega_n)} \qquad .....(2.151)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G(s)G_c(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_d s)}{s\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)}}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)+K_p\omega_n^2\,(1+T_d s)}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+2\zeta\omega_n s+K_p\omega_n^2+K_p\omega_n^2 T_d s}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+(2\zeta\omega_n+K_p\omega_n^2 T_d)\,s+K_p\omega_n^2} \qquad \boxed{K_d = K_p T_d}$$

$$= \frac{\omega_n^2\,(K_p+K_d s)}{s^2+(2\zeta\omega_n+K_d\omega_n^2)\,s+K_p\omega_n^2} \qquad .....(2.152)$$

From the closed loop transfer function (equation (2.152)) it is observed that the PD-controller introduces a zero in the system and increases the damping ratio. The addition of the zero may increase the peak overshoot and reduce the rise time. But the effect of increased damping ultimately reduces the peak overshoot.

From the loop transfer function (equation (2.151)) it is observed that the PD-controller does not modify the type number of the system. Hence PD-controller will not act modify steady state error.

EFFECT OF PID-CONTROLLER

A suitable combination of the three basic modes : *proportional, integral and derivative* (PID) can improve all aspects of the system performance.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error. The combined effect of all the three cannot be judged from the parameters K_p, K_i and K_d.

2.22 TIME RESPONSE ANALYSIS USING MATLAB

In general, the closed loop transfer function of a system is denoted as M(s).

Let, M(s) be a rational function of "s", as shown below.

$$M(s) = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + ... + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + ... + a_{N-1} s + a_N}$$

For time response analysis, the coefficients of the numerator and denominator polynomials are declared as two arrays as shown below.

```
num_cof = [b0 b1 b2 ........ bM];
den_cof = [a0 a1 a2 ........ aN];
```

UNIT STEP RESPONSE

To compute step response

The unit step response can be computed and displayed using following commands.

```
syms s complex;
R = 1/s;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
S = R*M;
disp('Unit step response of the system is,');
step_res = ilaplace(S)
```

To plot step response

Method 1 :

The unit step response can be plotted using the following command.

```
step(num_cof, den_cof);
```

Method 2 :

The unit step response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = step(num_cof, den_cof,t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

The unit step response can be computed "n" times by varying some parameter of the system (coefficient / damping ratio / natural frequency of oscillation) using the following commands.

```
t = t_start : t_step : t_end ;
for i = 1 : n
     .
     .
c(1:k, i) = step(num_cof, den_cof,t);
     .
     .
end
plot(t,c,'k');
     where, c is an array where the values of response are stored.
             k is the number of samples of response to be computed.
```

Method 3 :

The unit step response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
sr = step(M,t);
plot(t,sr,'k');
```

IMPULSE RESPONSE

To compute impulse response

The impulse response can be computed and displayed using following commands.

```
syms s complex;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
disp('Unit step response of the system is,');
imp_res = ilaplace(S)
```

To plot impulse response

Method 1 :

The impulse response can be plotted using the following command.

```
impulse(num_cof, den_cof);
```

Method 2 :

The impulse response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
m = impulse(num_cof, den_cof,t);
plot(t,m,'k');
     where, m is an array where the values of impulse response are stored.
```

Method 3 :

The impulse response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
imp = impulse(M,t);
plot(t,imp,'k');
```

RESPONSE FOR ARBITRARY INPUT

The response of a system for an arbitrary input, r(t) can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = Lsim(num_cof, den_cof, r, t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

PROGRAM 2.1

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of damping ratio, ζ. Take, natural frequency of oscillation, ω_n=1 rad/sec.

```
%Unit step response for various values of damping ratio, zeta.
%The natural frequency of oscillation, wn=1.

clc
t=0:0.2:12;                                  %specify a time vector
c=zeros(61,6);                               %initialize response array as zero
zeta=[0 0.2 0.4 0.6 0.8 1];                 %store zeta as an array
     for n=1:6;                              %for loop to compute c(t) 6 times
          num_cof=[0 0 1];
          den_cof=[1 2*zeta(n) 1];
          c(1:61,n)=step(num_cof,den_cof,t);
     end

plot(t,c,'k'); grid
xlabel('time,t in sec'); ylabel('Unit step response,c(t)');

text(2.8,1.86,'\zeta=0')
text(2.8,1.58,'\zeta=0.2')
text(2.8,1.30,'\zeta=0.4')
text(2.8,1.12,'\zeta=0.6')
text(2.8,0.95,'\zeta=0.8')
text(2.8,0.72,'\zeta=1.0')
```

OUTPUT

The output waveforms are shown in fig p2.1.

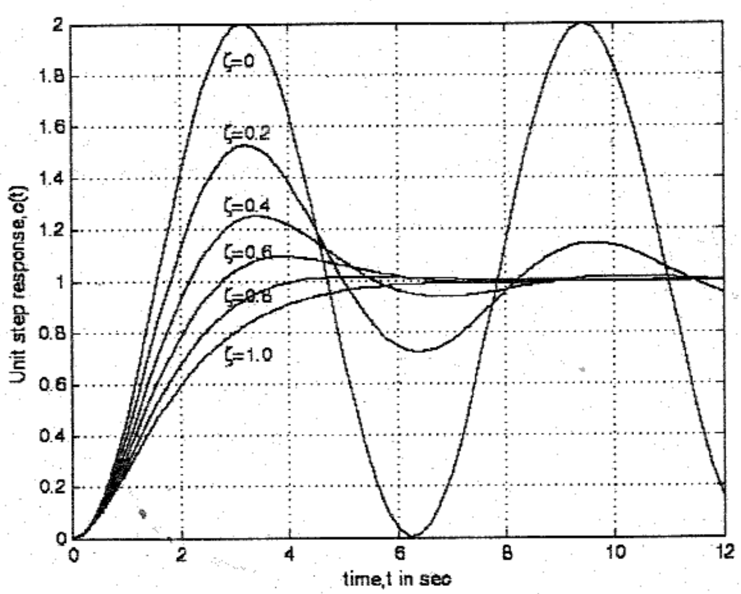


Fig P2.1 : Unit step response of second order system for various values of damping ratio.

PROGRAM 2.2

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of natural frequency of oscillation, ω_n. Take, damping ratio, $\zeta=0.4$.

```
%Unit step response for various natural frequency of oscillation,wn.
%The damping ratio, zeta=0.4.
clc
t=0:0.1:8;                  %specify a time vector
wn=[1 2 4 6];               %store wn as an array
zeta=0.4;
c=zeros(81,4);              %initialize the response array as zeros

   for i=1:4;               %for loop to compute c(t) 4 times
   b2=wn(i)*wn(i);
   a1=2*zeta*wn(i);
   num_cof=[0 0 b2];
   den_cof=[1 a1 b2];
   c(1:81,i)=step(num_cof,den_cof,t);
   end

plot(t,c(:,1),'--k',t,c(:,2),'xk',t,c(:,3),'-k',t,c(:,4),'-.k');
grid; xlabel('time,t in sec'); ylabel('Unit step response,c(t)');
text(4.25,1.25,'wn=1')
text(1.5,1.30,'wn=2')
text(0.7,1.30,'wn=3')
text(0.1,1.25,'wn=4')
```

OUTPUT

The output waveforms are shown in fig p2.2.

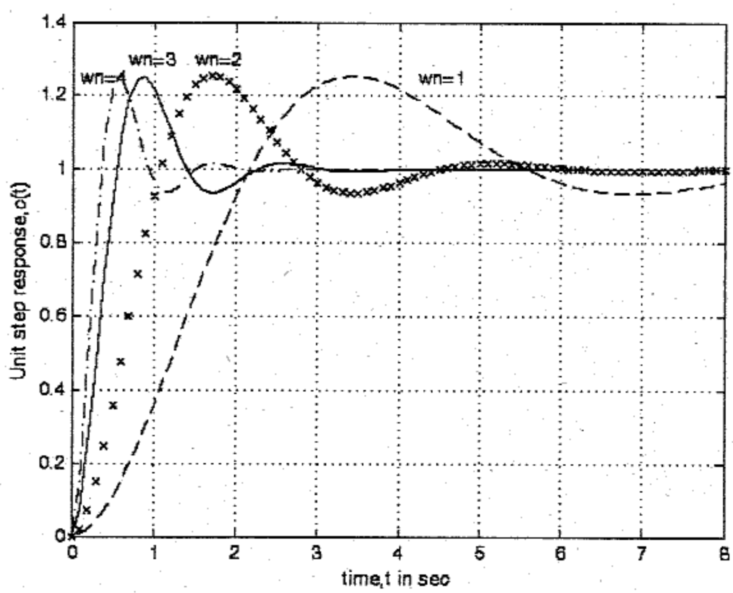


Fig P2.2 : Unit step response of second order system for various values of natural frequency of oscillation.

PROGRAM 2.3

Write a MATLAB program to find impulse response of the following systems.

a) $M_1(s)=(2s+1)/(s+1)^2$ b) $M_2(s)=s/(s+1)$ c) $M_3(s)=1/(s^2+1)$

```
%Program to find impulse response
clc
syms s complex;
M1=(2*s+1)/((s+1)^2);
disp('Impulse response of the system1 is,');
m1=ilaplace(M1)

M2=s/(s+1);
disp('Impulse response of the system2 is,');
m2=ilaplace(M2)

M3=1/(s^2+1);
disp('Impulse response of the system3 is,');
m3=ilaplace(M3)

s=tf('s');
M1=(2*s+1)/((s+1)^2);
M2=s/(s+1);
M3=1/(s^2+1);

t=0:.005:10;
m1=impulse(M1,t);
m2=impulse(M2,t);
m3=impulse(M3,t);

plot(t,m1,'--k',t,m2,'-.k',t,m3,'-k');grid
xlabel('time,t in sec');
ylabel('Impulse responses,m1(t),m2(t),m3(t)');
text(0.4,1.30,'m1(t)')
text(0.3,-0.30,'m2(t)')
text(2.2,0.90,'m3(t)')
```

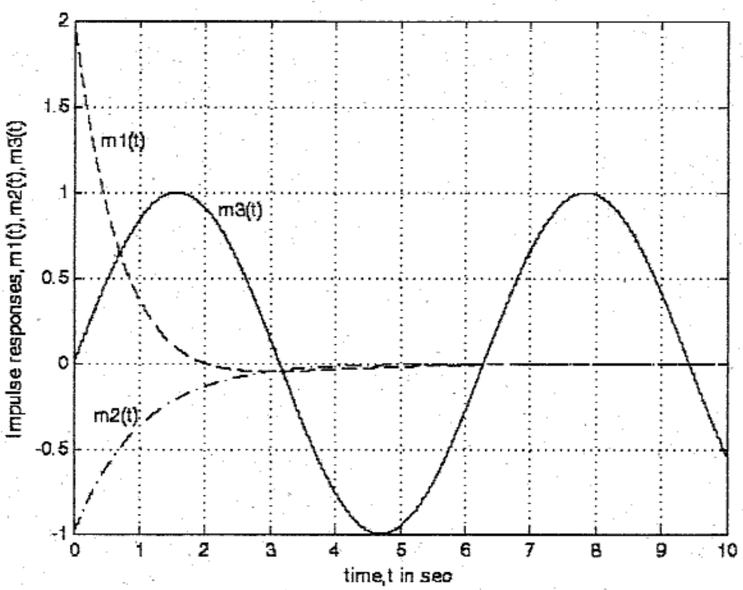


Fig P2.3 : Impulse response of systems given in program 2.3.

OUTPUT

```
Impulse response of the system1 is,
        m1 =
             (2-t)*exp(-t)

Impulse response of the system2 is,
        m2 =
              dirac(t)-exp(-t)

Impulse response of the system3 is,
        m3 =
             sin(t)

The output waveforms are shown in fig p2.3.
```

PROGRAM 2.4

Write a MATLAB program to find unit step response of the following systems.

a) $M_1(s)=4/(s^2+5s+4)$ b) $M_2(s)=100/(s^2+12s+100)$ c) $M_3(s)=600/(s^2+70s+600)$

```
%program to find unit step response
clc
syms s complex;
R=1/s;                          %Laplace of unit step input
M1=4/(s^2+5*s+4);
S1=R*M1;                        %s-domain unit step response of system1
disp('Unit step response of the system1 is,');
s1=ilaplace(S1)               %time domain unit step response of system1

M2=100/(s^2+12*s+100);
S2=R*M2;                        %s-domain unit step response of system2
disp('Unit step response of the system2 is,');
s2=ilaplace(S2)               %time domain unit step response of system2

M3=600/(s^2+70*s+600);
S3=R*M3;                        %s-domain unit step response of system3
disp('Unit step response of the system3 is,');
s3=ilaplace(S3)               %time domain unit step response of system3

s=tf('s');
M1=4/(s^2+5*s+4);
M2=100/(s^2+12*s+100);
M3=600/(s^2+70*s+600);

t=0:.005:10;
s1=step(M1,t);
s2=step(M2,t);
s3=step(M3,t);

plot(t,s1,'--k',t,s2,'-.k',t,s3,'-k');grid
xlabel('time,t in sec');
ylabel('Unit step responses,s1(t),s2(t),s3(t)');
text(2.2,0.85,'s1(t)')
text(0.2,1.15,'s2(t)')
text(0.5,0.95,'s3(t)')
```

OUTPUT

```
Unit step response of the system1 is,
       s1 =
               1/3*exp(-4*t)+1-4/3*exp(-t)

Unit step response of the system2 is,
       s2 =
               1-exp(-6*t)*cos(8*t)-3/4*exp(-6*t)*sin(8*t)

Unit step response of the system3 is,
       s3 =
               1+1/5*exp(-60*t)-6/5*exp(-10*t)

The output waveform is shown in fig p2.4.
```

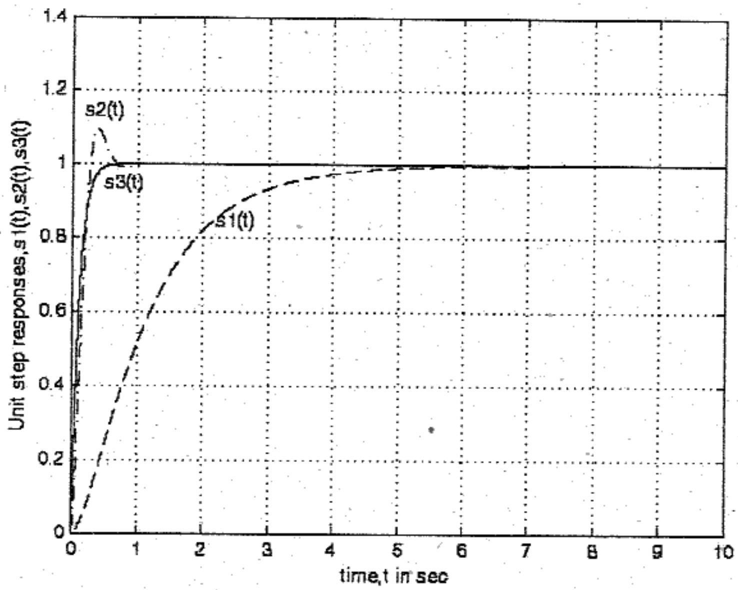


Fig P2.4 : Unit step response of systems given in program 2.4.

PROGRAM 2.5

Consider the closed loop transfer function of the following second order system,

$M(s)=16/(s^2+4s+16)$

Write a MATLAB program to find the rise time, peak time, maximum peak overshoot, and settling time from the unit step response of the system.

```
clc
t=0:0.005:5;                %set time vector

num_cof=[0 0 16];           %store the numerator coefficients as an array
den_cof=[1 4 16];           %store denominator coefficients as an array
[c,x,t]=step(num_cof,den_cof,t);

n=1;                        %initialize count as 1
while c(n)<1.0001;          %count the time index as along as c(t)<1
    n=n+1;
end;
```

```
rise_time=(n-1)*0.005   %rise time=(count-1)*time interval

[cmax,tp]=max(c);        %determine maximum value of c(t) &
                          %corresponding time
peak_time=(tp-1)*0.005 %peak time=(tp-1)*time interval

max_overshoot=cmax-1    %compute peak overshoot

n=1001;                    %initialize count as (5/.005)+1=1001
while c(n)>0.95&c(n)<1.05;
     n=n-1;                %count time index between c(t)>0.95&c(t)<1.05
end;
settling_time_5per_err=(n-1)*0.005

n=1001;                    %initialize count as (5/.005)+1=1001
while c(n)>0.98 & c(n)<1.02;
     n=n-1;                %count time index between c(t)>0.98&c(t)<1.02
end;
settling_time_2per_err=(n-1)*0.005
```

OUTPUT

```
rise_time =
              0.6050

peak_time =
              0.9050

max_overshoot =
              0.1630

settling_time_5per_err =
                     1.3200

settling_time_2per_err =
                     2.0150
```

PROGRAM 2.6

Consider the closed loop transfer function of the following second order system,

$M(s)=64/(s^2+8s+64)$

Write a MATLAB program to find the response for unit step, unit ramp and unit parabolic input signals.

```
%unit step/ramp/parabolic response

clc
num_cof=[0 0 64];
den_cof=[1 8 64];

t=0:0.005:2;

r1=t;              %unit ramp input signal
r2=0.5*t.^2;     %unit parabolic input signal

c1=step(num_cof, den_cof,t);
c2=Lsim(num_cof, den_cof,r1,t);
c3=Lsim(num_cof, den_cof,r2,t);
```

```
plot(t,c1,'--k',t,c2,'-.k',t,c3,'-k'); grid
xlabel('time,t in sec');
ylabel('Responses,c1(t),c2(t),c3(t)');
text(0.25,1.15,'c1(t)')
text(1.45,1.5,'c2(t)')
text(1.35,0.7,'c3(t)')
```

OUTPUT

The output waveform is shown in fig p2.6.

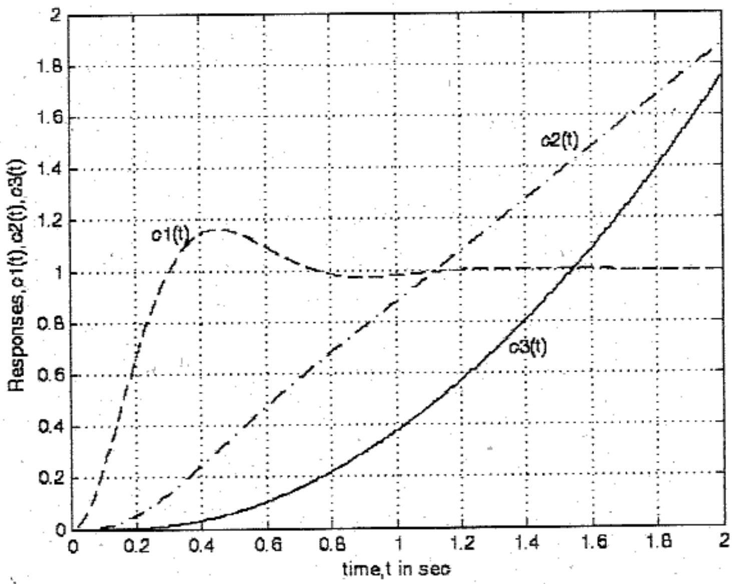


Fig P2.6 : *Step, ramp and parabolic response of system given in program 2.6.*

PROGRAM 2.7

Consider the closed loop transfer function of the following second order system,

$M(s)=5/(s^2+s+5)$

Write a MATLAB program to find the response for the input signal, $r(t)=2-2t+t^2$.

```
%program to find response for given input
clc
num_cof=[0  0  5];
den_cof=[1  1  5];
t=0:0.005:3;                            %specify a time vector

r=2-2*t+t.^2;                           %input signal

c=Lsim(num_cof,den_cof,r,t);    %compute response using Lsim function

plot(t,r,'--k',t,c,'-.k'); grid
xlabel('time,t in sec');
ylabel('Input,r(t) and output,c(t)');

text(0.25,1.65,'r(t)')
text(0.25,0.6,'c(t)')
```

OUTPUT

The output waveform is shown in fig p2.7.

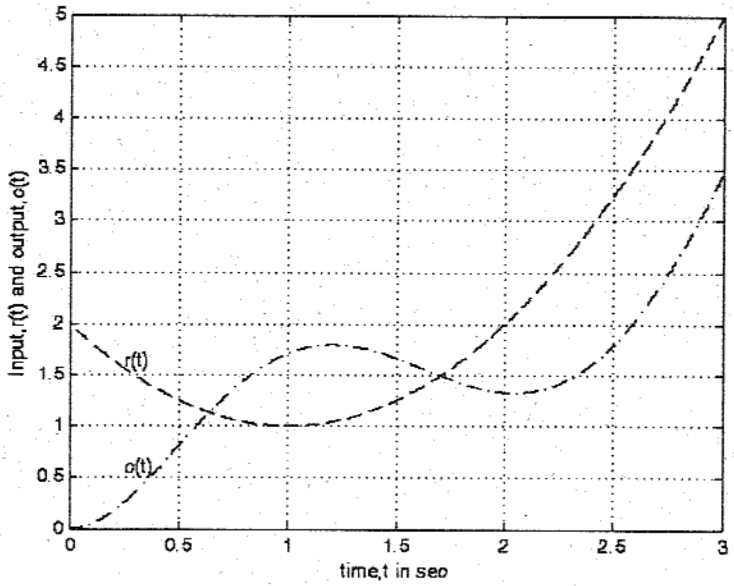


Fig P2.7 : Input and Output of the system given in program 2.7.

2.23 SHORT QUESTIONS AND ANSWERS

Q2.1 ***What is time response?***

The time response is the output of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse Laplace of the product of input and transfer function of the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Response in s-domain, $C(s) = \frac{R(s)\,G(s)}{1+G(s)\,H(s)}$

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)\,G(s)}{1+G(s)\,H(s)}\right\}$

Q2.2 ***What is transient and steady state response?***

The transient response is the response of the system when the input changes from one state to another. The response of the system as $t \to \infty$ is called steady state response.

Q2.3 ***What is the importance of test signals?***

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals. The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

Q2.4 ***Name the test signals used in control system.***

The commonly used test input signals in control system are Impulse, Step, Ramp, Acceleration and Sinusoidal signals.

Q2.5 ***Define step signal.***

The step signal is a signal whose value changes from 0 to A and remains constant at A for t > 0. The mathematical representation of step signal is,

$$r(t) = A,\ t \geq 0$$
$$= 0,\ t < 0$$

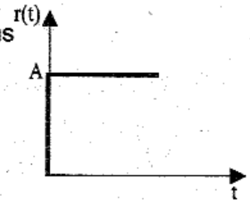


Fig Q2.5 : Step signal.

Q2.6 ***Define ramp signal.***

A ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. Mathematical representation of ramp signal is,

$$r(t) = At,\ t \geq 0$$
$$= 0,\ t < 0$$

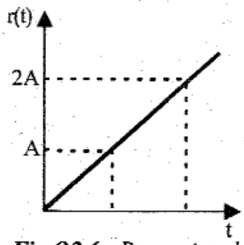


Fig Q2.6 : Ramp signal.

Q2.7 ***Define parabolic signal.***

It is a signal in which the instantaneous value varies as square of the time from an initial value of zero at t = 0.The mathematical representation of parabolic signal is,

$$r(t) = \frac{At^2}{2},\ t \geq 0$$
$$= 0,\quad t < 0$$

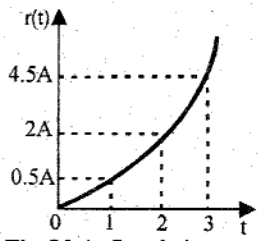


Fig Q2.4 : Parabolic signal.

Q2.8 ***What is weighing function?***

The impulse response of system is called weighing function. It is given by inverse Laplace transform of system transfer function.

Q2.9 ***What is an impulse signal?***

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at t = 0. At t = 0 the magnitude becomes infinite. It is denoted by δ(t) and mathematically expressed as,

$$\delta(t) = \infty\ ;\quad t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$
$$= 0\ ;\quad t = 0$$

Q2.10 ***Define pole.***

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable s.

Q2.11 ***Define zero.***

The zero of a function, F(s) is the value at which the function, F(s) becomes zero, where F(s) is a function of complex variable s.

Q2.12 ***What is the order of a system?***

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

Q2.13 ***Define damping ratio.***

The damping ratio is defined as the ratio of actual damping to critical damping.

Q2.14 ***Give the expression for damping ratio of mechanical and electrical system.***

The damping ratio of second order mechanical translational system, $\zeta = \frac{B}{2\sqrt{MK}}$

The damping ratio of second order mechanical rotational system, $\zeta = \frac{B}{2\sqrt{JK}}$

The damping ratio of second order electrical system, $\zeta = \dfrac{R}{2\sqrt{L/C}}$

Q2.15 ***How the system is classified depending on the value of damping?***

Depending on the value of damping, the system can be classified into the following four cases.

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

Q2.16. ***Sketch the response of a second order under damped system.***

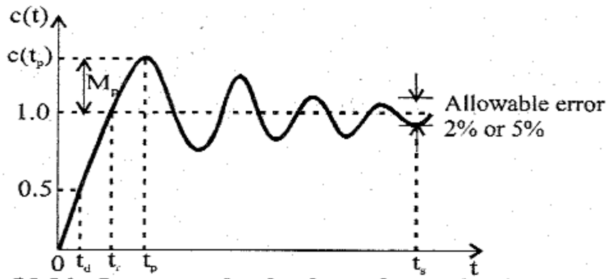


Fig Q2.16 : *Response of under damped second order system.*

Q2.17 ***What will be the nature of response of a second order system with different types of damping?***

For undamped system the response is oscillatory.
For underdamped system the response is damped oscillatory.
For critically damped system the response is exponentially rising.
For overdamped system the response is exponentially rising but the rise time will be very large.

Q2.18. ***What is damped frequency of oscillation?***

In underdamped system the response is damped oscillatory. The frequency of damped oscillation is given by, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Q2.19. ***Give the expression for natural frequency of oscillations of electrical and mechanical system.***

The natural frequency of oscillation of second order mechanical translational system $\Big\}\ \omega_n = \sqrt{\dfrac{K}{M}}$

The natural frequency of oscillation of second order mechanical rotational system $\Big\}\ \omega_n = \sqrt{\dfrac{K}{J}}$

The natural frequency of oscillation of second order electrical system $\Big\}\ \omega_n = \dfrac{1}{\sqrt{LC}}$

Q2.20. ***The closed loop transfer function of second order system is*** $\dfrac{C(s)}{R(s)} = \dfrac{10}{s^2+6s+10}$. ***What is the type of damping in the system?.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{10}{s^2+6s+10}$$

$\omega_n^2 = 10$	$2\zeta\omega_n = 6$
$\therefore \omega_n = \sqrt{10} = 3.1622$ rad/sec	$\therefore \zeta = \dfrac{6}{2\times\omega_n} = \dfrac{6}{2\times\sqrt{10}} = 0.95$

Since $\zeta < 1$, the system is underdamped.

Q2.21 ***The closed loop transfer function of a second order system is given by $\frac{200}{s^2+20s+200}$. Determine the damping ratio and natural frequency of oscillation.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{200}{s^2+20s+200}$$

$\therefore \omega_n^2 = 200$

$\omega_n = \sqrt{200} = 14.14 \text{ rad/sec}$

$2\zeta\omega_n = 20$

$\zeta = \frac{20}{2\times\omega_n} = \frac{20}{2\times 14.14} = 0.707$

Damping ratio, $\zeta = 0.707$

Natural frequency of oscillation, $\omega_n = 14.14$ rad/sec.

Q2.22 ***A second order system has a damping ratio of 0.6 and natural frequency of oscillation is 10 rad/sec. Determine the damped frequency of oscillation.***

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.6)^2} = 10\times 0.8 = 8 \text{ rad/sec}$

Q2.23 ***The open loop transfer function of a unity feedback system is $G(s) = \frac{20}{s(s+10)}$. What is the nature of response of closed loop system for unit step input.***

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/s(s+10)}{1+\frac{20}{s(s+10)}} = \frac{20}{s(s+10)+20} = \frac{20}{s^2+10s+20}$$

The standard form of second order transfer function is, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

On comparing system transfer function with standard form of second order transfer function we get,

$\omega_n^2 = 20$

$\therefore \omega_n = \sqrt{20} = 4.47 \text{ rad/sec}$

$2\zeta\omega_n = 10$

$\zeta = \frac{10}{2\times\omega_n} = \frac{10}{2\times 4.47} = 1.12$

Since damping ratio, $\zeta > 1$, the system is overdamped and the response will be exponentially rising.

Q2.24 ***List the time domain specifications.***

The time domain specifications are,

(i) Delay time (ii) Rise time (iii) Peak time

(iv) Maximum overshoot (v) Settling time.

Q2.25 ***Define delay time.***

It is the time taken for response to reach 50% of the final value, the very first time.

Q2.26 ***Define rise time.***

It is the time taken for response to raise from 0 to 100% , the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

Q2.27 ***Define peak time.***

It is the time taken for the response to reach the peak value, the very first time (or) It is the time taken for the response to reach peak overshoot, M_p.

Q2.28 ***Define peak overshoot.***

It is defined as the ratio of the maximum peak value to final value, where maximum peak value is measured from final value.

Let final value = $c(\infty)$, Maximum value = $c(t_p)$ $\therefore$ Peak overshoot, $M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}$.

Q2.29 *Define settling time.*

It is defined as the time taken by the response to reach and stay within a specified error and the error is usually specified as % of final value. The usual tolerable error is 2% or 5% of the final value.

Q2.30 *The damping ratio of a system is 0.75 and the natural frequency of oscillation is 12 rad/sec. Determine the peak overshoot and the peak time.*

Peak overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.75\times\pi}{\sqrt{1-(0.75)^2}}} = 0.028$; $\therefore \%M_p = 0.028 \times 100 = 2.8\%$

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 12\sqrt{1-(0.75)^2} = 7.94 \text{ rad/sec}$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.94} = 0.396 \text{ sec}$

Q2.31 *The damping ratio of system is 0.6 and the natural frequency of oscillation is 8 rad/sec. Determine the rise time.*

Rise time, $t_r = \frac{\pi-\theta}{\omega_d}$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 53.13° = \frac{53.13}{180}\times\pi \text{ rad} = 0.927 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8\sqrt{1-(0.6)^2} = 6.4 \text{ rad/sec}$$

$\therefore$ Rise time, $t_r = \frac{\pi-0.927}{6.4} = 0.34 \text{ sec}$

Q2.32 *What is type number of a system? What is its significance?*

The type number is given by number of poles of loop transfer function at the origin. The type number of the system decides the steady state error.

Q2.33 *Distinguish between type and order of a system.*

(i) Type number is specified for loop transfer function but order can be specified for any transfer function. (open loop or closed loop transfer function).

(ii) The type number is given by number of poles of loop transfer function lying at origin of s-plane but the order is given by the number of poles of transfer function.

Q2.34 *For the system with following transfer function, determine type and order of the system.*

(i) $G(s)\,H(s) = \frac{K}{s(s+1)\,(s^2+6s+8)}$ (ii) $G(s)\,H(s) = \frac{20\,(s+2)}{s^2(s+3)\,(s+0.5)}$

(iii) $G(s)\,H(s) = \frac{(s+4)}{(s-2)\,(s+0.25)}$ (iv) $G(s)\,H(s) = \frac{10}{s^3(s^2+2s+1)}$

Ans: (i) Type - 1, order - 4 (ii) Type - 2, order - 4

(iii) Type - 0, order - 2 (iv) Type - 3, order - 5.

Q2.35 *What is steady state error?*

The steady state error is the value of error signal e(t), when t tends to infinity . The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Q2.36 *What are static error constants?*

The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

Q2.37 *Define positional error constant.*

The positional error constant $K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s)$ .The steady state error in type-0 system when the input is unit step is given by$1/1+K_p$.

Q2.38 ***Define velocity error constant.***

The velocity error constant $K_v = \underset{s \to 0}{Lt}\ sG(s)\,H(s)$. The steady state error in type-1 system for unit ramp input is given by $1/K_v$.

Q2.39 ***Define acceleration error constant.***

The acceleration error constant $K_a = \underset{s \to 0}{Lt}\ s^2\,G(s)\,H(s)$. The steady state error in type-2 system for unit parabolic input is given by $1/K_a$.

Q2.40 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{10}{(s+1)\,(s+2)}$. Determine the steady state error for unit step input.***

The steady state error for unit step input, $e_{ss} = \dfrac{1}{1+K_p}$, where, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$.

For unity feedback system H(s) = 1.

$$\therefore K_p = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt}\ \frac{10}{(s+1)\,(s+2)} = 5 \qquad \text{and} \qquad e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$$

Q2.41 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{25\,(s+4)}{s(s+0.5)\,(s+2)}$. Determine the steady state error for unit ramp input.***

The steady state error for unit ramp input is, $e_{ss} = \dfrac{1}{K_v}$, where, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_v = \underset{s \to 0}{Lt}\ sG(s) = \underset{s \to 0}{Lt}\ s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25 \times 4}{0.5 \times 2} = 100 \qquad \text{and} \qquad e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

Q2.42 ***A unity feedback system has a open loop transfer function of $G(s) = \dfrac{20\,(s+5)}{s(s+0.1)\,(s+3)}$. Determine the steady state error for parabolic input.***

The steady state error for unit ramp input is $e_{ss} = \dfrac{1}{K_a}$, where, $K_a = \underset{s \to 0}{Lt}\ s^2G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_a = \underset{s \to 0}{Lt}\ s^2\left[\frac{20\,(s+5)}{s^2(s+0.1)\,(s+3)}\right] = \frac{20 \times 5}{0.1 \times 3} = \frac{100}{0.3} = 333.33 \qquad \text{and} \qquad e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

Q2.43 ***What are generalized error coefficients?***

They are the coefficients of generalized error series.The generalized error series is given by,

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \dots + \frac{C_n}{n!}\overset{n}{r}(t)\dots$$

The coefficients $C_0, C_1, C_2 \dots\dots C_n$ are called generalized error coefficients or dynamic error coefficients.

$$\text{The } n^{th} \text{ coefficient, } C_n = \underset{s \to 0}{Lt}\ \frac{d^n}{ds^n}F(s), \quad \text{where, } F(s) = \frac{1}{1+G(s)\,H(s)}.$$

Q2.44 ***Give the relation between generalized and static error coefficients.***

The following expression shows the relation between generalized and static error coefficient.

$$C_0 = \frac{1}{1+K_p}; \qquad C_1 = \frac{1}{K_v}; \qquad C_2 = \frac{1}{K_a}$$

Q2.45 ***Mention two advantages of generalized error constants over static error constants.***

(i) Generalized error series gives error signal as a function of time.

(ii) Using generalized error constants the steady state error can be determined for any type of input but static error constants are used to determine steady state error when the input is anyone of the standard input.

Q2.46. *What are the basic components of an automatic control system ?*

The basic components of an automatic control system are,

1. Error detector
2. Amplifier and controller
3. Actuator (Power actuator)
4. Plant
5. Sensor or feedback system

Q2.47 *What is automatic controller ?*

The combined unit of error detector, amplifier and controller is called automatic controller.

Q2.48 *What is the need for a controller?*

The controller is provided to modify the error signal for better control action.

Q2.49 *What are the different types of controllers?*

The different types of controller used in control system are P, PI, PD and PID controllers.

Q2.50 *What is Proportional controller and what are its advantages?*

The Proportional controller is a device that produces a control signal which is proportional to the input error signal.

The advantages in the proportional controller are improvement in steady-state tracking accuracy disturbance signal rejection and the relative stability. It also makes a system less sensitive to parameter variations.

Q2.51 *What is the drawback in P-controller?*

The drawback in P-controller is that it develop a constant steady-state error.

Q2.52 *What is integral control action?*

In integral control action, the control signal is proportional to integral of error signal.

Q2.53 *What is the advantage and disadvantage in integral controller?*

The advantage in Integral controller is that it eliminates or reduces the steady-state error. The disadvantage is that it can make a system unstable.

Q2.54 *Write the transfer function of P, PI, PD and PID controllers.*

The transfer function of P-controller, $\frac{U(s)}{E(s)} = K_p$; where, K_p = Proportional gain.

The transfer function of PI-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right)$; where, T_i = Integral time constant.

The transfer function of PD-controller, $\frac{U(s)}{E(s)} = K_p(1+T_d s)$; where, T_d = Derivative time constant.

The transfer function of PID-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$

Q2.55 *What is Reset rate?*

The Reset rate is the reciprocal of integral time or reset time. The reset rate is the number of times per minute that the proportional part of the control action is duplicated and it is measured in terms of repeats/minute.

Q2.56 *Why derivative control is not employed in isolation?*

A derivative control mode in isolation produces no corrective efforts for any constant errors. Because it acts only on rate of change of error.

Q2.57 *What is PI-controller?*

The PI-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Q2.58 What is PD-controller?

The PD-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

Q2.59 What is PID-Controller?

The PID-controller is a device which produces a control signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

Q2.60 Give an example of electronic PID-controller

The electronic PID-controller can be realized by an op-amp amplifier with integral and derivative action followed by sign changer, as shown in figure Q2.60.

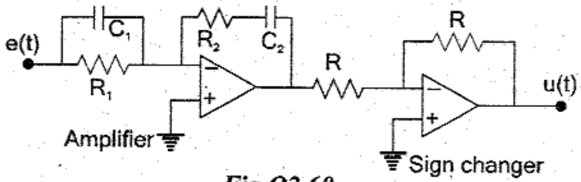


Fig Q2.60

Q2.61 Sketch the step response of a P and PI-controller ?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.61.

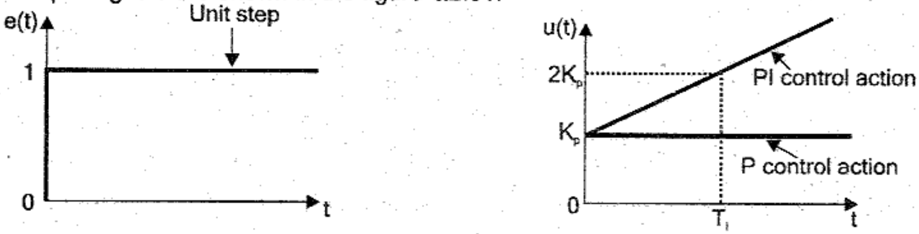


Fig Q2.61

Q2.62 Sketch the ramp response of P, PD and PID-controller?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.62.

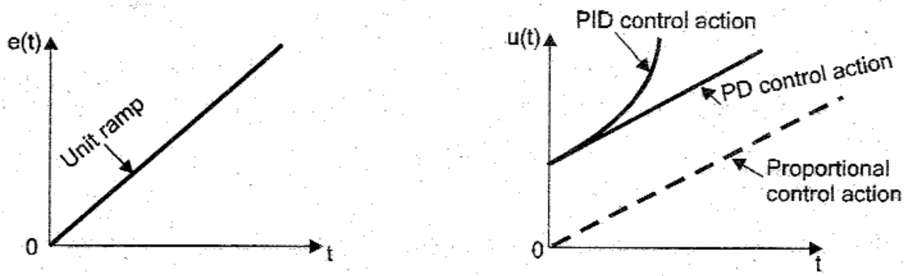


Fig Q2.62

Q2.63 What is the effect on system performance when a proportional controller is introduced in a system?

The proportional controller improves the steady-state tracking accuracy, disturbance signal rejection and relative stability of the system. It also increases the loop gain of the system which results in reducing the sensitivity of the system to parameter variations.

Q2.64 What is the disadvantage in proportional controller?

The disadvantage in proportional controller is that it produces a constant steady state error.

Q2.65 ***What is the effect of PI-controller on the system performance?***

The PI - controller increases the order of the system by one, which results in reducing, the steady state error. But the system becomes less stable than the original system.

Q2.66 ***What is the effect of PD-controller on the system performance?***

The effect of PD - controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

Q2.67 ***Why derivative controller is not used in control systems?***

The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control systems.

Q2.68 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \dfrac{2s+1}{(s+1)^2}$.

By partial fraction expansion the given closed loop transfer function can be expressed as,

$$\therefore M(s) = \frac{2s+1}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

$$A = \left.\frac{2s+1}{(s+1)^2} \times (s+1)^2\right|_{s=-1} = 2s+1\big|_{s=-1} = 2(-1)+1 = -1$$

$$B = \left.\frac{d}{ds}\left[\frac{2s+1}{(s+1)^2} \times (s+1)^2\right]\right|_{s=-1} = \left.\frac{d}{ds}[2s+1]\right|_{s=-1} = 2$$

$$\therefore M(s) = \frac{-1}{(s+1)^2} + \frac{2}{(s+1)}$$

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^2} + \frac{2}{s+1}\right\} = -t\,e^{-t} + 2\,e^{-t}$$

Q2.69 ***Determine the impulse response of the feedback systems governed by the following closed loop transfer functions,***

a) $M(s) = \dfrac{s}{s+1}$; **b)** $M(s) = \dfrac{1}{s^2+1}$.

a) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{1}{s+1}\right\} = \delta(t) - e^{-t}$$

b) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

Q2.70 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \dfrac{2(s+3)}{(s+3)^2+1}$.

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+1}\right\} = 2\,e^{-3t}\cos t$$

2.24 EXERCISES

E2.1 What is the unit-step response of the system shown in fig E2.1

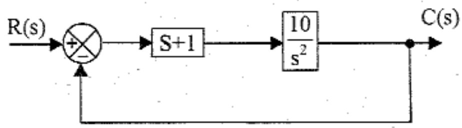


Fig E2.1.

E2.2 Obtain the unit-step response of a unity-feedback system whose open-loop transfer function is $G(s)=\dfrac{5(s+20)}{s(s+4.59)\,(s^2+3.41s+16.35)}$.

E2.3 The open loop transfer function of an unity feedback control system is given by $G(s)=\dfrac{100}{s(s+2)\,(s+5)}$.

For unit step input,find the time response of the closed loop system and determine % over shoot and the rise time.

E2.4 A Servomechanism has its moment of inertia J = 10×10^{-6} Kg-m², retarding friction, B = 400×10^{-6} N-m/(rad/sec) and elasticity coefficient, K = 0.004 N-m/rad. Find the natural frequency and damping factor of the system.

E2.5 For a second order system whose open loop transfer function $G(s)=\dfrac{4}{s(s+2)}$, determine the maximum over shoot, the time to reach the maximum overshoot when a step displacement of 18° is given to the system. Find the rise time, time constant and the settling time for an error of 7% .

E2.6 Consider the unity feedback closed loop system where the forward transfer function is $G(s)=\dfrac{25}{s(s+5)}$.

Obtain the rise time, Peak time, Maximum overshoot and the settling time when the system is subjected to a unit-step input.

E2.7 Consider the system shown in fig E2.7, where $\zeta = 0.6$ and $\omega_n = 0.5$ rad/sec. Determine the rise time, peak time, maximum overshoot and settling time, when the system is subjected to a unit-step input.

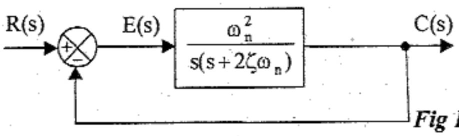


Fig E2.7.

E2.8 For the system shown in fig E2.8, determine the values of K and K_h so that the maximum overshoot in the unit step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain rise time and settling time.

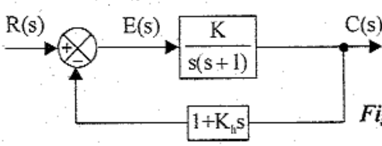


Fig E2.8.

E2.9 The system shown in fig E2.9 subjected to a unit-step input. Determine the values of K and T, where the Maximum overshoot of the system is 25.4% corresponding to $\zeta = 0.4$.

E2.10 Determine the values of K and T of the closed-loop system shown in Fig E2.10, so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 Kg-m².

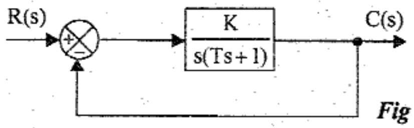


Fig E2.9.

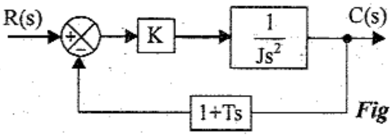


Fig E2.10.

E2.11 A unity-feedback system is characterized by the open-loop transfer function $G(s)=\frac{1}{s(0.5s+1)\,(0.2s+1)}$.

a) Determine the steady-state errors to unit-step, unit-ramp and unIt parabolic inputs.

b) Determine rise time, peak time, peak overshoot and settling time of the unit-step respone of the system.

E2.12 For a system whose $G(s)=\frac{10}{s(s+1)\,(s+2)}$,
find the steady state error when it is subjected to the input, $r(t) = 1+2t+1.5\,t^2$.

E2.13 A unity feedback system has $G(s)=\frac{1}{s(1+s)}$.
The input to the system is described by $r(t) = 4+6t+2t^3$. Find the generalized error coefficients and steady state error.

E2.14 A unity feedback system has the forward path transfer function $G(s)=\frac{10}{(s+1)}$.

Find the steady state error and the generalised error coefficient for $r(t) = t$.

E2.15 Find out the position, velocity and acceleration error coefficients for the following unity feedback systems having forward loop transfer function G(s) as,

(a) $\frac{100}{(1+0.5s)\,(1+2s)}$ (b) $\frac{K}{s(1+0.1s)\,(1+s)}$

(c) $\frac{K}{s^2(s^2+8s+100)}$ (d) $\frac{K(1+s)\,(1+2s)}{s^2(s^2+4s+20)}$

E2.16 The open loop transfer function of a unity feedback control system is $G(s) = 9/(s+1)$, using the generalized error series determine the error signal and steady state error of the system when the system is excited by,

(i) $r(t)=2$ (ii) $r(t)=t$

(iii) $r(t)=3t^2/2$ (iv) $r(t)=1+2t+3t^2/2$

E2.17 For unity feedback system having open loop transfer function as $G(s)=\frac{K(s+2)}{s^2(s^2+7s+12)}$. Determine,

(i) type of system,

(ii) error constants K_p, K_v and K_a

(iii) steady state error for parabolic input.

ANSWER FOR EXERCISE PROBLEMS

E2.1 $c(t) = -1.1455\, e^{-8.87t} + 0.1455 e^{-1.13t} + 1$

E2.2 $c(t) = 1 + \frac{3}{8} e^{-t} \cos 3t - \frac{17}{24} e^{-t} \sin 3t - \frac{11}{8} e^{-3t} \cos t - \frac{13}{8} e^{-3t} \sin t$

E2.3 $c(t) = \left[1 - 0.186\, e^{-7.45t} - 0.88\, e^{0.225t} \cos(3.65t - 22°)\right]$

As t tends to infinity, c(t) tends to infinity and so the system is unstable. Therefore % over shoot and rise time are not defined.

E2.4 Natural frequency, ω_n = 20 rad/sec, Damping factor, ζ = 1.

E2.5 Maximum overshoot = 0.16, when input is 18%, M_p = 2.88%
Peak time, t_p=1.81sec. Rise time, t_r=1.21 sec
Time constant, T=1 sec, Settling time for 7% error = 2.66 sec.

E2.6 Rise time, t_r = 0.55sec, %Peak overshoot, M_p = 9.5%
Peak time, t_p = 0.785sec, Settling time, t_s = 1.33sec (for 2% error); t_s =1 sec(for 5% error)

E2.7 Rise time, t_r = 0.55sec, Maximum overshoot, M_p = 0.095
Peak time, t_p = 0.785sec, Settling time, t_s = 1 sec (for 5% criterion)

E2.8 K=12.5, Rise time, t_r = 0.65sec
K_h = 0.178 ; Settling time, t_s = 2.48 sec (for 2% error) ; t_s = 1.86 sec (for 5% error)

E2.9 K=1.42, T=1.09

E2.10 K=2.95 N-m T=0.471 sec

E2.10 (a) $e_{ss}\big|_{\text{unit step}} = 0$

$e_{ss}\big|_{\text{unit ramp}} = 1$

$e_{ss}\big|_{\text{unit parabola}} = \infty$

(b) Rise time, $t_r = 1.91$ sec
Peak time, $t_p = 2.79$ sec
Peak overshoot, $M_p = 0.1265$
Settling time, $t_s = 5.4$ sec

E2.12 The total steady state error is ∞.

E2.13 $C_0 = 0$; $C_1 = 1$; $C_2 = 0$; $C_3 = -6$; $e_{ss} = \infty$

E2.14 $C_0 = 1/11$; $C_1 = 10/121$; $e_{ss} = \infty$.

E2.15

Question	K_p	K_v	K_a
(a)	100	0	0
(b)	∞	K	0
(c)	∞	∞	K/100
(d)	∞	∞	K/20

E2.16 (i) e(t) = 0.2 ; e_{ss} = 0.2
(ii) e(t) = 0.1t+0.09 ; $e_{ss} = \infty$.
(iii) $e(t) = 0.15t^2 + 0.27\,t - 0.054$; $e_{ss} = \infty$.
(iv) $e(t) = 0.15t^2 + 0.77\,t + 0.226$; $e_{ss} = \infty$.

E2.17 (i) It is type-2 system
(ii) $K_p = \infty$; $K_v = \infty$; $K_a = K/6$
(iii) $e_{ss} = 6/K$

CHAPTER 3

FREQUENCY RESPONSE ANALYSIS

3.1 SINUSOIDAL TRANSFER FUNCTION AND FREQUENCY RESPONSE

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response and sinusoidal input is called ***sinusoidal transfer function*** of the system and in general, it is denoted by $T(j\omega)$. The sinusoidal transfer function is the frequency domain representation of the system, and so it is also called ***frequency domain transfer function.***

The sinusoidal transfer, function $T(j\omega)$ can be obtained as shown below.

1. Contruct a physical model of a system using basic elements/parameters.
2. Determine the differential equations governing the system from the physical model of the system.
3. Take Laplace transform of differential equations in order to convert them to s-domain equation.
4. Determine s-domain transfer function, T(s), which is ratio of s-domain output and input.
5. Determine the frequency domain transfer function, $T(j\omega)$ by replacing s by $j\omega$ in the s-domain transfer function, T(s).

> *Note : If the s-domain transfer function, T(s) is known, then frequency domain transfer function, $T(j\omega)$ can be obtained directly from T(s) by replacing s by $j\omega$.*
>
> i.e., $T(s) \xrightarrow{s=j\omega} T(j\omega)$

Consider a linear time invariant system with frequency domain transfer function, $T(j\omega)$ shown in fig 3.1. Let the system be excited by a sinusoidal signal frequency ω, amplitude A, and phase θ. Now the response or output will also be a sinusoidal signal of same frequency ω, but the amplitude and phase of response will be modified by amplitude and phase of the transfer function respectively.

Now, the amplitude of the response is given by the product of the amplitude of the input and transfer function. The phase of the response is given by the sum of the phase of the input and transfer function.

Let, $T(j\omega) = |T(j\omega)| \angle T(j\omega)$

where, $|T(j\omega)|$ = Magnitude of $T(j\omega)$, and, $\angle T(j\omega)$ = Phase of $T(j\omega)$.

Let, Input, $r(t) = A \sin(\omega t + \theta) = A \angle\theta$

where, A = Amplitude of input, ω = Frequency of input, and θ = Phase of input.

Now, Response, $c(t) = r(t) \times T(j\omega) = A\angle\theta \times |T(j\omega)| \angle T(j\omega) = A \times |T(j\omega)| \angle(\theta + \angle T(j\omega)) = B\angle\phi$

where, $B = A \times |T(j\omega)|$ = Magnitude of response, and, $\phi = \theta + \angle T(j\omega)$ = Phase of response.

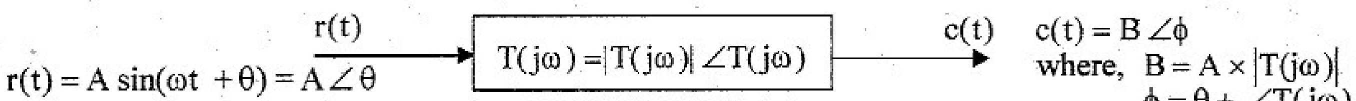


Fig 3.1 : System with sinusoidal transfer function $T(j\omega)$.

FREQUENCY RESPONSE

The frequency domain transfer function T(jω) is a complex function of ω. Hence it can be seperated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω, and they are called ***frequency response.***

The frequency response can be evaluated for open loop system and closed loop system. The frequency domain transfer function of open loop and closed loop systems can be obtained from the s-domain transfer function by replacing s by jω shown below.

Open loop transfer function : $G(s) \xrightarrow{s=j\omega} G(j\omega) = |G(j\omega)| \angle G(j\omega)$ .....(3.1)

Loop transfer function : $G(s)H(s) \xrightarrow{s=j\omega} G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$ .....(3.2)

Closed loop transfer function: $M(s) \xrightarrow{s=j\omega} M(j\omega) = |M(j\omega)| \angle M(j\omega)$ .....(3.3)

where, $|G(j\omega)|$, $|M(j\omega)|$, $|G(j\omega)H(j\omega)|$ are Magnitude functions
$\angle G(j\omega)$, $\angle M(j\omega)$, $\angle G(j\omega)H(j\omega)$ are Phase functions.

Note : For unity feedback system, H(s) = 1 and open loop and loop transfer functions are same.

The advantages of frequency response analysis are the following.

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments .
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.
4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by use of the frequency response analysis, the effects of noise disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain nonlinear control systems.

3.2 FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are,

1. Resonant peak , M_r
2. Resonant Frequency , ω_r
3. Bandwidth, ω_b
4. Cut-off rate
5. Gain margin, K_g
6. Phase margin, γ

Resonant Peak (M_r)

The maximum value of the magnitude of closed loop transfer function is called the resonant peak, M_r. A large resonant peak corresponds to a large overshoot in transient response.

Resonant Frequency (ω_r)

The frequency at which the resonant peak occurs is called resonant frequency, ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Bandwidth (ω_b)

The Bandwidth is the range of frequencies for which normalized gain of the system is more than –3 db. The frequency at which the gain is –3 db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Cut-off Rate

The slope of the log-magnitude curve near the cut off frequency is called cut-off rate. The cut -off rate indicates the ability of the system to distinguish the signal from noise.

Gain Margin ,K_g

The gain margin, K_g is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability.

The gain margin, K_g is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency. The frequency at which the phase of open loop transfer function is 180° is called the phase cross-over frequency, ω_{pc}.

$$\text{Gain Margin, } K_g = \frac{1}{|G(j\omega_{pc})|} \qquad .....(3.4)$$

The gain margin in db can be expressed as,

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|} \qquad .....(3.5)$$

Note : $|G(j\omega_{pc})|$ *is the magnitude of* $G(j\omega)$ at $\omega = \omega_{pc}$

The Gain margin in db is given by the negative of the db magnitude of $G(j\omega)$ at phase cross-over frequency. The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

Phase Margin (γ)

The phase margin γ, is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability. The gain cross over frequency ω_{gc} is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin γ, is obtained by adding 180° to the phase angle ϕ of the open loop transfer function at the gain cross over frequency

$$\text{Phase margin, } \gamma = 180° + \phi_{gc}, \qquad .....(3.6)$$

$$\text{where, } \phi_{gc} = \angle G(j\omega_{gc})$$

Note : $\angle G(j\omega_{gc})$ *is the phase angle of* $G(j\omega)$ at $\omega = \omega_{gc}$

The phase margin indicates the additional phase lag that can be provided to the system without affecting stability.

3.3 FREQUENCY DOMAIN SPECIFICATIONS OF SECOND ORDER SYSTEM

RESONANT PEAK (M_r)

Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(3.7)$$

The sinusoidal transfer function M(jω) is obtained by letting s = jω.

$$\therefore M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \qquad .....(3.8)$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1\right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

Let, Normalized frequency, $u = \left(\frac{\omega}{\omega_n}\right)$

$$\therefore M(j\omega) = \frac{1}{(1-u^2) + j2\zeta u}$$

Let, M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

$$M = |M(j\omega)| = \left[\frac{1}{(1-u^2)^2 + (2\zeta u)^2}\right]^{\frac{1}{2}} = \left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} \qquad .....(3.9)$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2} \qquad .....(3.10)$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting dM/du = 0 when $u = u_r$,

where, $u_r = \frac{\omega_r}{\omega}$ = Normalized resonant frequency.

On differentiating equation (3.9) with respect to u we get,

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} = -\frac{1}{2}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{3}{2}}\left[2(1-u^2)(-2u) + 8\zeta^2u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\zeta^2u\right]}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} = \frac{4u(1-u)^2 - 8\zeta^2u}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} \qquad .....(3.11)$$

Replace u by u_r in equation (3.11) and equate to zero.

$$\frac{4u_r(1-u_r^2) - 8\zeta^2u_r}{2\left[(1-u_r^2)^2 + 4\zeta^2u_r^2\right]^{\frac{3}{2}}} = 0 \qquad .....(3.12)$$

The equation (3.12) will be zero if numerator is zero. Hence, on equating numerator to zero we get,

$$4u_r(1-u_r^2)-8\zeta^2u_r=0 \quad \Rightarrow \quad 4u_r-4u_r^3-8\zeta^2u_r=0$$

$$\therefore\ 4u_r^3=4u_r-8\zeta^2u_r \quad \Rightarrow \quad u_r^2=1-2\zeta^2 \quad \Rightarrow \quad u_r=\sqrt{1-2\zeta^2} \qquad \text{......(3.13)}$$

Therefore, the resonant peak occurs when $u_r=\sqrt{1-2\zeta^2}$

Put this condition in the equation for M and solve for M_r.

$$\therefore M_r=\left.\frac{1}{\left[(1-u^2)^2+4\zeta^2u^2\right]^{\frac{1}{2}}}\right|_{u=u_r}=\frac{1}{\left[(1-u_r^2)^2+4\zeta^2u_r^2\right]^{\frac{1}{2}}}=\frac{1}{\left[(1-(1-2\zeta^2))^2+4\zeta^2(1-2\zeta^2)\right]^{\frac{1}{2}}}$$

$$=\frac{1}{\left[4\zeta^4+4\zeta^2-8\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2-4\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2(1-\zeta^2)\right]^{\frac{1}{2}}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore \boxed{\text{Resonant peak, } M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}}} \qquad \text{......(3.14)}$$

RESONANT FREQUENCY (ω_r)

Normalized resonant frequency, $u_r=\dfrac{\omega_r}{\omega_n}=\sqrt{1-2\zeta^2}$ .....(3.15)

The resonant frequency, $\omega_r=\omega_n\sqrt{1-2\zeta^2}$ .....(3.16)

BANDWIDTH (ω_b)

Let, Normalized bandwidth, $u_b=\dfrac{\omega_b}{\omega_n}$

When $u=u_b$, the magnitude M, of the closed loop system is $1/\sqrt{2}$ (or –3db).

Hence in the equation for M (equation 3.9), put $u=u_b$ and equate to $1/\sqrt{2}$.

$$\therefore\ M=\frac{1}{\left[(1-u_b^2)^2+4\zeta^2u_b^2\right]^{\frac{1}{2}}}=\frac{1}{\sqrt{2}} \qquad \text{.....(3.17)}$$

On squaring and cross multiplying we get,

$$(1-u_b^2)^2+4\zeta^2u_b^2=2 \quad \Rightarrow \quad 1+u_b^4-2u_b^2+4\zeta^2u_b^2=2 \quad \Rightarrow \quad u_b^4-2u_b^2(1-\zeta^2)-1=0$$

Let, $x=u_b^2$; $\quad \therefore x^2-2(1-2\zeta^2)x-1=0$

$$\therefore x=\frac{2(1-2\zeta^2)\pm\sqrt{4(1-2\zeta^2)^2+4}}{2}=\frac{2(1-2\zeta^2)\pm2\sqrt{(1+4\zeta^4-4\zeta^2)+1}}{2}$$

Let us take only the positive sign,

$$\therefore x=1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}$$

But, $u_b=\sqrt{x}$; $\quad \therefore u_b=\sqrt{x}=\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{\frac{1}{2}}$; $\quad$ Also, $u_b=\dfrac{\omega_b}{\omega_n}$

$$\therefore \boxed{\text{Bandwidth, } \omega_b = \omega_n u_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{\frac{1}{2}}} \qquad .....(3.18)$$

PHASE MARGIN (γ)

The open loop transfer function of second order system,

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \qquad .....(3.19)$$

The sinusoidal transfer function $G(j\omega)$ is obtained by letting $s = j\omega$.

$$G(j\omega) = \frac{\omega_n^2}{j\omega\,(j\omega + 2\zeta\omega_n)} = \frac{\omega_n^2}{\omega_n\left(\frac{j\omega}{\omega_n}\right)\omega_n\left(2\zeta + j\frac{\omega}{\omega_n}\right)} = \frac{1}{j\frac{\omega}{\omega_n}\left(2\zeta + j\frac{\omega}{\omega_n}\right)} \qquad .....(3.20)$$

Let normalized frequency, $u = \omega/\omega_n$

On substituting $u = \omega/\omega_n$ in equation (3.20) we get,

$$G(j\omega) = \frac{1}{ju(2\zeta + ju)} \qquad .....(3.21)$$

$$\text{Magnitude of } G(j\omega) = |G(j\omega)| = \frac{1}{u\sqrt{4\zeta^2 + u^2}} = \frac{1}{\sqrt{u^4 + 4\zeta^2 u^2}} \qquad .....(3.22)$$

$$\text{Phase of } G(j\omega) = -90^\circ - \tan^{-1}\frac{u}{2\zeta} \qquad .....(3.23)$$

At the gain cross-over frequency ω_{gc}, the magnitude of $G(j\omega)$ is unity.

Let normalized gain cross over frequency, $u_{gc} = \omega_{gc}/\omega_n$

On substituting u by u_{gc} in the equation (3.22) and equating to unity, we get,

$$\therefore \text{At } u = u_{gc},\ |G(j\omega)| = \frac{1}{\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2}} = 1 \quad \Rightarrow \quad u_{gc}^4 + 4\zeta^2 u_{gc}^2 = 1 \quad \Rightarrow \quad u_{gc}^4 + 4\zeta^2 u_{gc}^2 - 1 = 0$$

Let, $\mathbf{x = u_{gc}^2}$; $\quad \therefore x^2 + 4\zeta^2 x - 1 = 0$

$$\therefore x = \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 + 4}}{2} = -2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

Let us take only the positive sign,

$$\therefore x = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

$$\text{But, } u_{gc} = \sqrt{x}\ ; \quad \therefore u_{gc} = \sqrt{x} = \left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}} \qquad .....(3.24)$$

$$\text{The phase margin, } \gamma = 180 + \angle G(j\omega)\big|_{\omega = \omega_{gc},\, u = u_{gc}} \qquad .....(3.25)$$

Substituting for $\angle G(j\omega)$ from equation (3.23) in equation (3.25) we get,

$$\gamma = 180 + \left(-90^\circ - \tan^{-1}\frac{u_{gc}}{2\zeta}\right) = 90 - \tan^{-1}\left[\frac{\left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}}}{2\zeta}\right] \qquad .....(3.26)$$

Note : The gain margin of second order system is infinite.

3.4 CORRELATION BETWEEN TIME AND FREQUENCY RESPONSE

The correlation between time and frequency response has an explicit form only for first and second order systems. The correlation for second-order system is discussed here.

Consider the magnitude and phase of a closed loop second order system as a function of normalized frequency, as given by equations (3.9) and (3.10).

Magnitude of closed loop system, $M = |M(j\omega)| = \dfrac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}$

Phase of closed loop system, $\alpha = \angle M(j\omega) = -\tan^{-1}\dfrac{2\zeta u}{1-u^2}$

The magnitude and phase angle characteristics for normalized frequency u, for certain values of ζ are shown in fig 3.2 and 3.3. The frequency at which M has a peak value is known as the resonant frequency. The peak value of the magnitude is the resonant peak M_r. At this frequency the slope of the magnitude curve is zero. The frequency corresponding to M_r is u_r, which is the normalized resonant frequency.

From equations (3.14) and (3.15) we get,

Resonant peak, $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

When $\zeta = 0$, $\quad \omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n$ .....(3.27)

When $\zeta = 0$, $\quad M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = \infty$ .....(3.28)

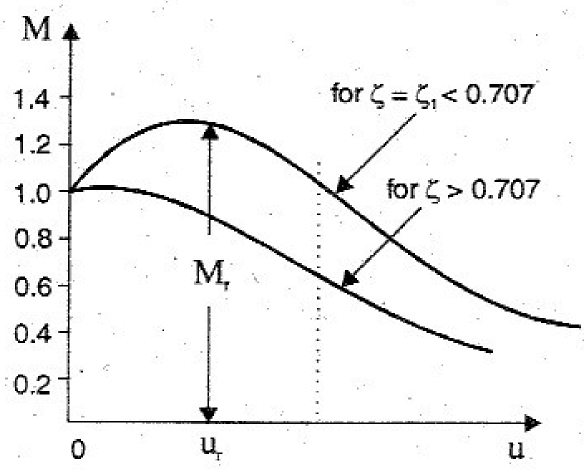


Fig 3.2 : Magnitude, M as a function of u.

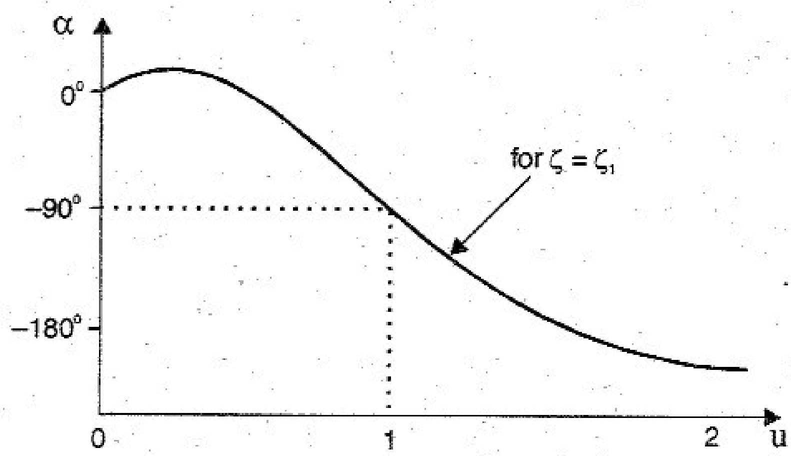


Fig 3.3 : Phase, α *as a function of u.*

From equations (3.27) and (3.28), it is clear that as ζ tends to zero, ω_r approaches ω_n, and M_r approaches infinity.

When $1-2\zeta^2 = 0$, $\omega_r = 0$, which means there is no resonant peak at this condition.

Let, $1-2\zeta^2 = 0$; $\therefore \zeta^2 = \dfrac{1}{2} \quad \Rightarrow \quad \zeta = \dfrac{1}{\sqrt{2}}$

For $0 < \zeta \le 1/\sqrt{2}$, the resonant frequency always has a value less than ω_n, and the resonant peak has a value greater than one.

For $\zeta > 1/\sqrt{2}$, the condition (dM/du) = 0, will not be satisfied for any real value of ω.

Hence when $\zeta > 1/\sqrt{2}$ the magnitude M decreases monotonically from M = 1 at u = 0 with increasing u. It follows that for $\zeta > 1/\sqrt{2}$ there is no resonant peak and the greatest value of M equals one.

The frequency at which M has a value of $1/\sqrt{2}$ is of special significance and is called the cut-off frequency ω_c. The signal frequencies above cut-off are greatly attenuated on passing through a system.

For feedback control system, the range of frequencies over which $M \geq 1/\sqrt{2}$ is defined as bandwidth ω_b. Control system being low-pass filters (at zero frequency M = 1), the bandwidth ω_b is equal to cut-off frequency ω_c.

In general the bandwidth of a control system indicates the noise-filtering characteristics of the system. Also, bandwidth gives a measure of the transient response.

The normalized bandwidth, $u_b = \frac{\omega_b}{\omega_n} = \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}\right]^{\frac{1}{2}}$

From the equation of u_b it is clear that u_b is a function of ζ alone. The graph between u_b and ζ is shown in fig 3.4.

The expression for the damped frequency of oscillation ω_d and peak overshoot M_p of the step response, for $0 \leq \zeta \leq 1$ are,

Damped frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and Peak overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

Comparison of the equation of M_r and M_p reveals that both are functions of only ζ.

The sketch of M_r and M_p for various value of ζ are shown in fig 3.5. The sketches reveals that a system with a given value of M_r must exhibit a corresponding value of M_p if subjected to a step input. For $\zeta > 1/\sqrt{2}$, the resonant peak M_r does not exist and the correlation breaks down. This is not a serious problem as for this range of ζ, the step response oscillations are well damped and M_p is negligible.

The comparison of the equation of ω_r and ω_d reveals that there exists a definite correlation between them. The sketch of ω_r/ω_d with respect to ζ is shown in fig 3.6.

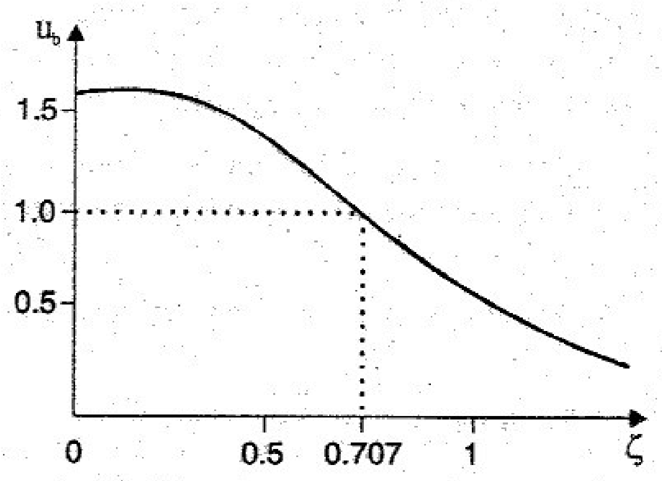


Fig 3.4 : Normalised bandwidth as a function of ζ.

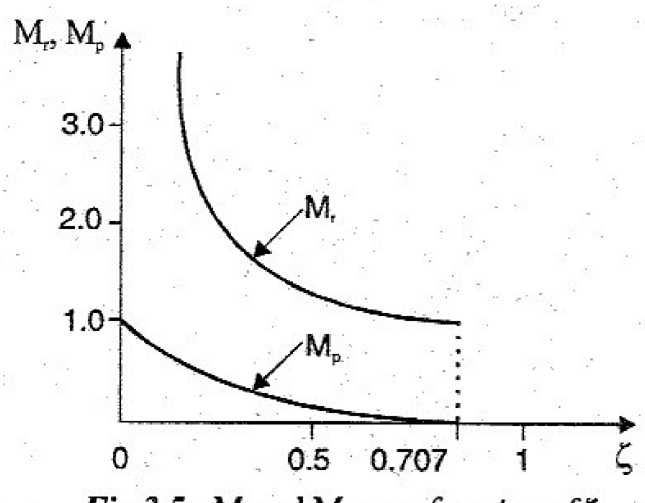


Fig 3.5 : M_r and M_p as a function of ζ.

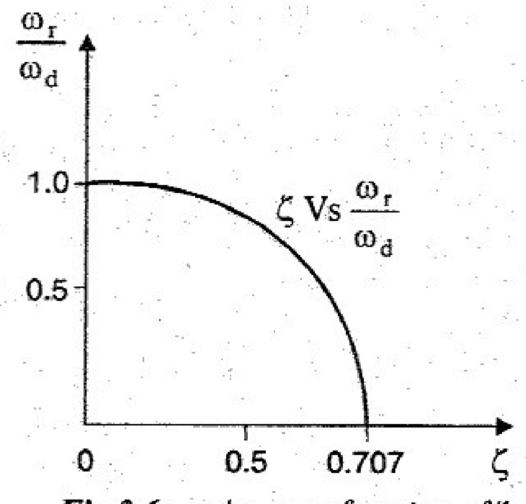


Fig 3.6 : ω_r/ω_d as a function of ζ.

3.5 FREQUENCY RESPONSE PLOTS

Frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are,

1. Bode plot
2. Polar plot (or Nyquist plot)
3. Nichols plot
4. M and N circles
5. Nichols chart

The Bode plot, Polar plot and Nichols plot are usually drawn for open loop systems. From the open loop response plot, the performance and stability of closed loop system are estimated. The M and N circles and Nichols chart are used to graphically determine the frequency response of unity feedback closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specifications, to study the stability of the systems and to adjust the gain of the system to satisfy the desired specifications.

3.6 BODE PLOT

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. A Bode plot consists of two graphs. One is a plot of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω.

The Bode plot can be drawn for both open loop and closed loop system. Usually the bode plot is drawn for open loop system. The standard representation of the logarithmic magnitude of open loop transfer function of G(jω) is 20 log |G(jω)| where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel,usually abbreviated as db. The curves are drawn on semilog paper, using the log scale (abcissa) for frequency and the linear scale (ordinate) for either magnitude (in decibels) or phase angle (in degrees).

The main advantage of the bode plot is that multiplication of magnitudes can be converted into addition. Also a simple method for sketching an approximate log-magnitude curve is available.

Consider the open loop transfer function, $G(s) = \dfrac{K\,(1+sT_1)}{s\,(1+sT_2)\,(1+sT}$

$$G(j\omega) = \frac{K\,(1+j\omega T_1)}{j\omega\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ\ \sqrt{1+\omega^2T_1^2}\ \angle\tan^{-1}\omega T_1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_2^2}\ \angle\tan^{-1}\omega T_2\sqrt{1+\omega^2T_3^2}\ \angle\tan^{-1}\omega T_3}$$

The magnitude of $G(j\omega) = |G(j\omega)| = \dfrac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\ \sqrt{1+\omega^2T_3^2}}$

The phase angle of the $G(j\omega) = \angle G(j\omega) = \tan^{-1}\omega T_1 - 90^\circ - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$

The magnitude of G(jω) can be expressed in decibels as shown below.

$$|G(j\omega)| \text{ in db} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2T_1^2}}{\omega\sqrt{1+\omega^2T_2^2}\ \sqrt{1+\omega^2T_3^2}}\right.$$

$$= 20\log\left[\frac{K}{\omega}\times\sqrt{1+\omega^2T_1^2}\times\frac{1}{\sqrt{1+\omega^2T_2^2}}\times\frac{1}{\sqrt{1+\omega^2T_3^2}}\right]$$

$$= 20\log\frac{K}{\omega}+20\log\sqrt{1+\omega^2T_1^2}+20\log\frac{1}{\sqrt{1+\omega^2T_2^2}}+20\log\frac{1}{\sqrt{1+\omega^2T_3^2}}$$

$$= 20\log\frac{K}{\omega}+20\log\sqrt{1+\omega^2T_1^2}-20\log\sqrt{1+\omega^2T_2^2}-20\log\sqrt{1+\omega^2T_3^2} \quad .....(3.29)$$

From the equation (3.29) it is clear that, when the magnitude is expressed in db, the multiplication is converted to addition. Hence in magnitude plot, the db magnitudes of individual factors of $G(j\omega)$ can be added.

Therefore to sketch the magnitude plot, a knowledge of the magnitude variations of individual factor is essential. The magnitude plot and phase plot of various factors of $G(j\omega)$ are explained in the following section.

BASIC FACTORS OF $G(j\omega)$

The basic factors that very frequently occur in a typical transfer function $G(j\omega)$ are,

1. Constant gain, K
2. Integral factor, $\frac{K}{j\omega}$ or $\frac{K}{(j\omega)^n}$
3. Derivative factor, $K\times j\omega$ or $K\times(j\omega)^n$
4. First order factor in denominator, $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. First order factor in numerator, $(1+j\omega T)$ or $(1+j\omega T)^m$
6. Quadratic factor in denominator, $\left[\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. Quadratic factor in numerator, $\left[1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$

Constant Gain, K

Let, $G(s) = K$

$$\therefore G(j\omega) = K = K\angle 0^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20\log K$$

$$\phi = \angle G(j\omega) = 0^\circ$$

The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20 log K db. The phase plot is straight line at 0°.

When $K > 1$, 20 log K is positive.

When $0 < K < 1$, 20 log K is negative.

When $K = 1$, 20 log K is zero.

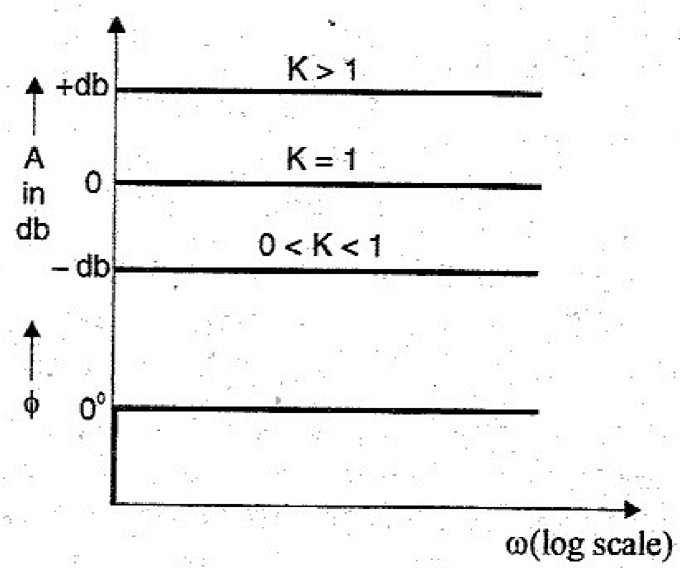


Fig 3.7 : Bode plot of constant gain, K.

Integral Factor

Let, $G(s) = \frac{K}{s}$

$$\therefore G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega}\angle -90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K/\omega)$$

$$\phi = \angle G(j\omega) = -90^\circ$$

When $\omega = 0.1$ K, $A = 20 \log (1/0.1) = 20$ db

When $\omega = K$, $A = 20 \log 1 = 0$ db

When $\omega = 10$ K, $A = 20 \log (1/10) = -20$ db

From the above analysis it is evident that the magnitude plot of the integral factor is a straight line with a slope of –20 db/dec and passing through zero db, when $\omega = K$. Since the $\angle G(j\omega)$ is a constant and independent of ω the phase plot is a straight line at –90°.

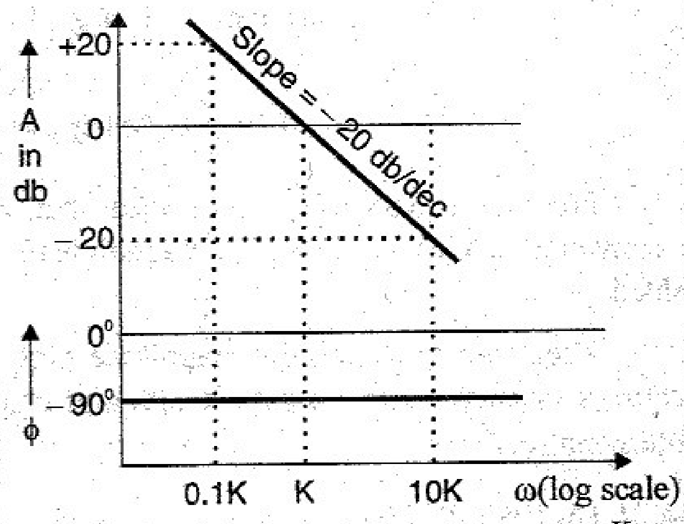


Fig 3.8 : Bode plot of integral factor, $\frac{K}{j\omega}$.

When an integral factor has multiplicity of n, then,

$$G(s) = K/s^n$$

$$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{K}{\omega^n}$$

$$= 20\log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)^n = 20\, n \log\left(\frac{K^{\frac{1}{n}}}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90\, n^\circ$$

Now the magnitude plot of the integral factor is a straight line with a slope of –20n db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at –90n°.

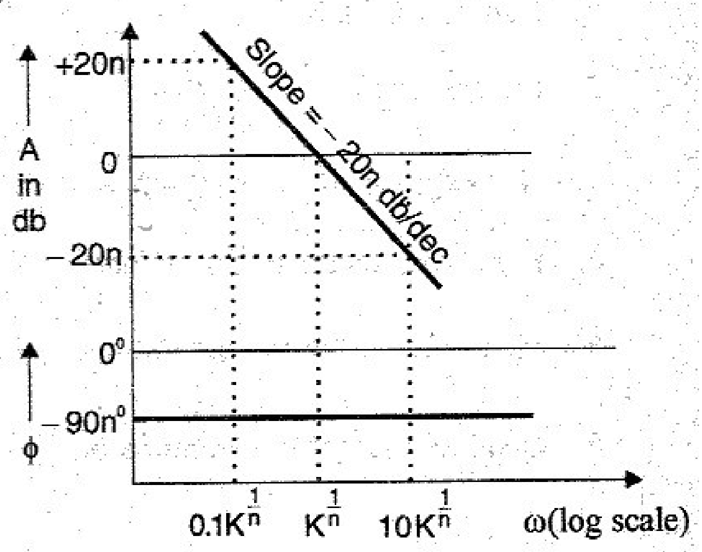


Fig 3.9 : Bode plot of integral factor, $K/(j\omega)^n$.

Derivative Factor

Let, $G(s) = Ks$

$$\therefore G(j\omega) = K\, j\omega = K\omega\ \angle 90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega)$$

$$\phi = \angle G(j\omega) = +90^\circ$$

When $\omega = 0.1/K$, $A = 20 \log (0.1) = -20$ db

When $\omega = 1/K$, $A = 20 \log 1 = 0$ db

When $\omega = 10/K$, $A = 20 \log 10 = +20$ db

From the above analysis it is evident that the magnitude plot of the derivative factor is a straight line with a slope of +20 db/dec and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at +90°.

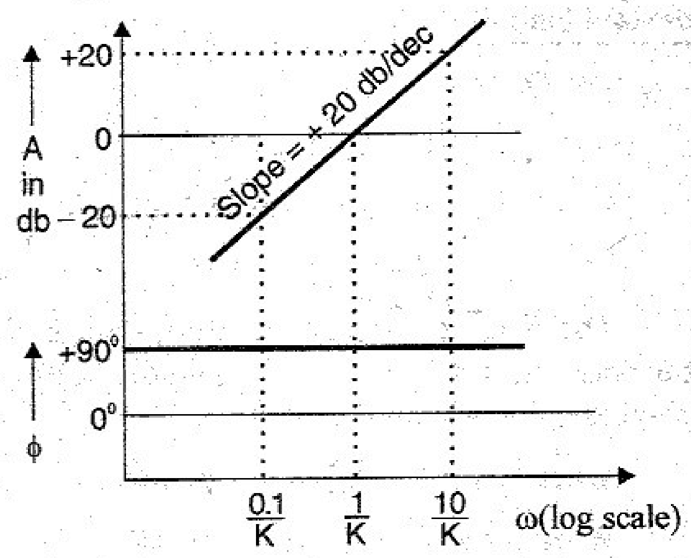


Fig 3.10 : Bode plot of derivative factor, $K \times j\omega$.

When derivative factor has multiplicity of n then,

$$G(s) = K\,s^n$$

$$\therefore G(j\omega) = K(j\omega)^n = K\omega^n \angle 90n°$$

$$A = |G(j\omega)| \text{ in db} = 20\log(K\omega^n)$$

$$= 20\log(K^{1/n}\,\omega)^n = 20\,n\log(K^{1/n}\,\omega)$$

$$\phi = \angle G(j\omega) = 90n°$$

Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at +90n°.

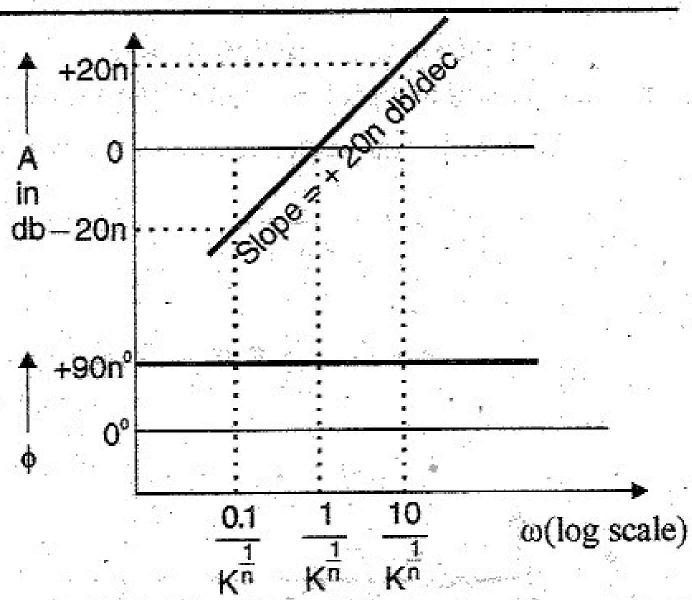


Fig 3.11 : Bode plot of derivative factor, $K(j\omega)^n$.

First order factor in denominator

$$G(s) = \frac{1}{1+sT}$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

Let, $A = |G(j\omega)|$ in db.

$$\therefore\ A = |G(j\omega)|_{\text{in db}} = 20\log\frac{1}{\sqrt{1+\omega^2T^2}} = -20\log\sqrt{1+\omega^2T^2}$$

At very low frequencies, $\omega T << 1$; $\quad \therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log 1 = 0$

At very high frequencies, $\omega T >> 1$; $\quad \therefore A = -20\log\sqrt{1+\omega^2T^2} \approx -20\log\sqrt{\omega^2T^2} = -20\log\omega T$

$$\text{At } \omega = \frac{1}{T},\quad A = -20\log 1 = 0$$

$$\text{At } \omega = \frac{10}{T},\quad A = -20\log 10 = -20\text{ db}$$

The above analysis shows that the magnitude plot of the factor $1/(1+j\omega T)$ can be approximated by two straight lines, one is a straight line at 0 db for the frequency range, $\mathbf{0 < \omega < 1/T}$, and the other is a straight line with slope –20 db/dec for the frequency range, $\mathbf{1/T < \omega < \infty}$. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called ***corner frequency*** or ***break frequency***. For the factor $1/(1+j\omega T)$ the frequency, $\omega = 1/T$ is the corner frequency, ω_c. It divides the frequency response curve into two regions, a curve for low frequency region and a curve for high frequency region.

The actual magnitude at the corner frequency, $\omega_c = \dfrac{1}{T}$ is,

$$A = -20\log\sqrt{1+1} = -3\text{ db.}$$

Hence by this approximation the loss in db at the corner frequency is –3 db.

The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω

$$\text{Phase angle, } \phi = \angle G(j\omega) = -\tan^{-1}\omega T$$

At the corner frequency, $\omega = \omega_c = \dfrac{1}{T}$, $\phi = -\tan^{-1}\omega T = -\tan^{-1}1 = -45°$

$$\text{As } \omega \to 0,\quad \phi \to 0°$$

$$\text{As } \omega \to \infty,\quad \phi \to -90°$$

The phase angle of the factor, $1/(1+j\omega T)$, varies from 0° to –90° as ω is varied from zero to infinity. The phase plot is a curve passing through –45° at ω_c.

When the first order factor in the denominator has a multiplicity of m, then,

$$G(s)=\frac{1}{(1+sT)^m}\ ; \qquad \therefore G(j\omega)=\frac{1}{(1+j\omega T)^m}=\frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T}$$

$$A=|G(j\omega)| \text{ in db}=20\log\frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m}=-20\, m\log\sqrt{1+\omega^2T^2}$$

$$\phi=\angle G(j\omega)=-m\tan^{-1}\omega T$$

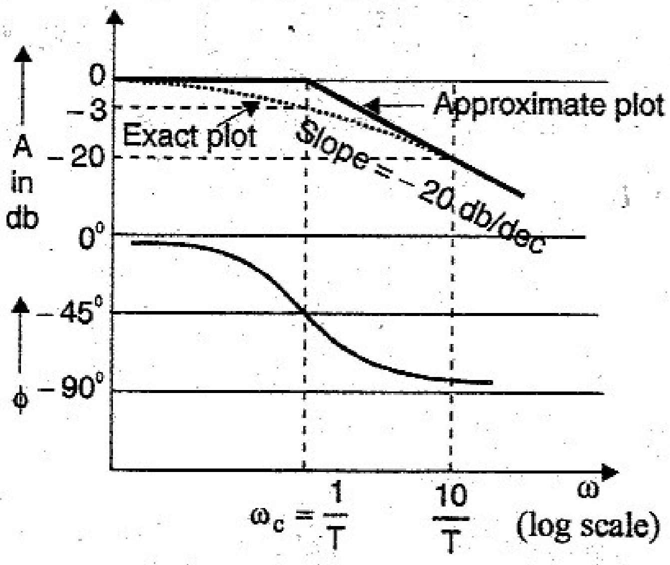


Fig 3.12 : Bode plot of the factor $\frac{1}{1+j\omega T}$.

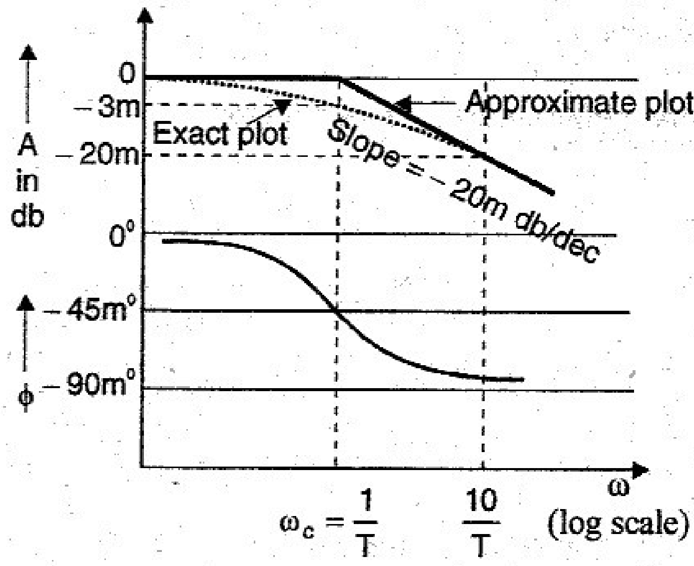


Fig 3.13 : Bode plot of the factor $1/(1+j\omega T)^m$.

Now the magnitude plot of the factor $1/(1+j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0<\omega<1/T$, and the other is a straight line with slope –20 m db/dec for the frequency range, $1/T<\omega<\infty$. The corner frequency, $\omega_c=1/T$ and the loss in db at the corner frequency is –3m db.

The phase angle of the factor $1/(1+j\omega T)^m$ varies from 0° to –90m° as ω is varied from zero to infinity. The phase plot is a curve passing through –45m° at ω_c.

FIRST ORDER FACTOR IN THE NUMERATOR

$$G(s)=1+sT$$

$$G(j\omega)=1+j\omega T=\sqrt{1+\omega^2T^2}\ \angle\tan^{-1}\omega T$$

$$A=|G(j\omega)| \text{ in db}=20\log\sqrt{1+\omega^2T^2}$$

$$\phi=\angle G(j\omega)=\tan^{-1}\omega T$$

By an analysis similar to that of previous section it can be shown that the magnitude plot of the factor $(1+j\omega T)$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0<\omega<1/T$ and the other is a straight line with slope +20 db/dec for the frequency range $1/T<\omega<\infty$. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called the corner frequency or break frequency. For the factor $(1+j\omega T)$, the frequency, $(\omega=1/T)$ is the corner frequency, ω_c. By this approximation the loss in db at the corner frequency is +3 db. The phase angle of the factor $(1+j\omega T)$ varies from zero to +90° as ω is varied from 0 to ∞. The phase plot is a curve passing through +45° at ω_c.

When the first order factor in the numerator has a multiplicity of m, then,

$$G(s) = (1+sT)^m$$

$$G(j\omega) = (1+j\omega T)^m = \left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\left(\sqrt{1+\omega^2T^2}\right)^m = 20m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = m\tan^{-1}\omega T$$

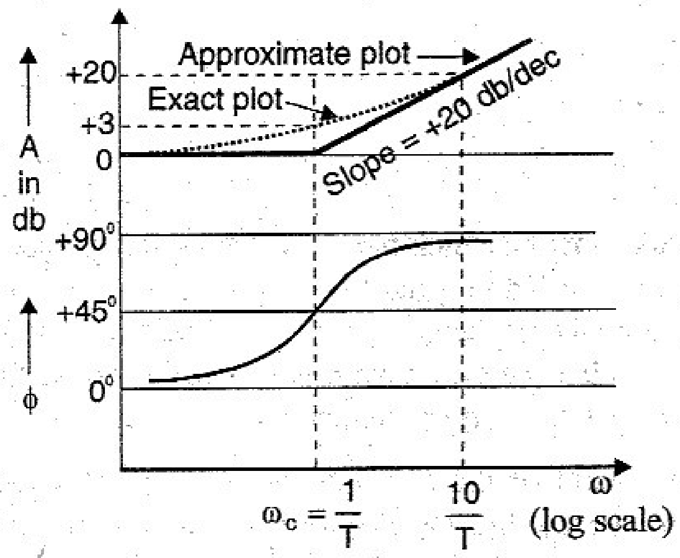


Fig 3.14 : Bode plot of the factor (1+jωT).

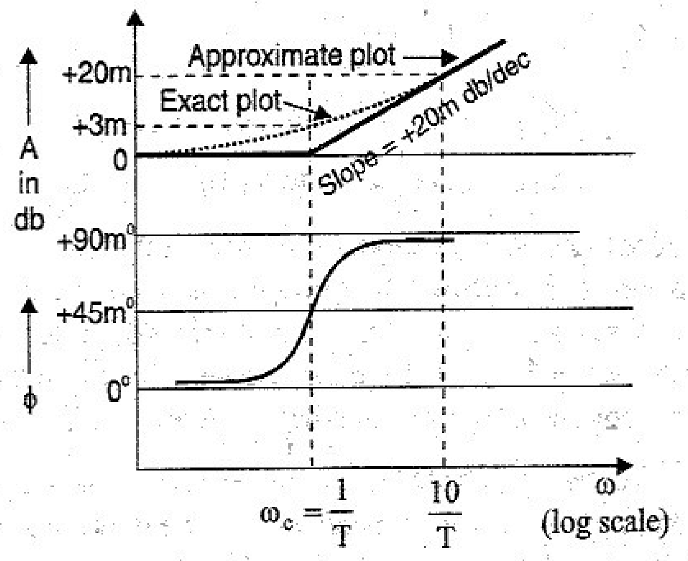


Fig 3.15 : Bode plot of the factor (1+jωT)^m.

Now the magnitude plot of the factor $(1 + j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with a slope of +20m db/dec for the frequency range $1/T < \omega < \infty$. The corner frequency, $\omega_c = 1/T$ and the loss in db at this corner frequency is +3m db.

The phase angle of the factor $(1 + j\omega T)^m$ varies from zero to +90m° as ω is varied from zero to infinity. The phase plot is a curve passing through +45 m° at ω_c.

QUADRATIC FACTOR IN THE DENOMINATOR

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{1+2\zeta\dfrac{s}{\omega_n}+\left(\dfrac{s}{\omega_n}\right)^2}$$

$$\therefore G(j\omega) = \frac{1}{1+j\dfrac{2\zeta\omega}{\omega_n}+\left(\dfrac{j\omega}{\omega_n}\right)^2} = \frac{1}{1-\left(\dfrac{\omega}{\omega_n}\right)^2+j2\zeta\dfrac{\omega}{\omega_n}} = \frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\dfrac{\omega^2}{\omega_n^2}}} \angle -\tan^{-1}\frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\dfrac{\omega^2}{\omega_n^2}}$$

Let, $A = |G(j\omega)|$ in db.

$$A = 20\log\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\dfrac{\omega^2}{\omega_n^2}}} = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$= -20\log\sqrt{1+\frac{\omega^4}{\omega_n^4}-2\frac{\omega^2}{\omega_n^2}+4\zeta^2\frac{\omega^2}{\omega_n^2}} = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}$$

At very low frequencies when $\omega << \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}} \approx -20\log 1 = 0$$

At very high frequencies when $\omega >> \omega_n$, the magnitude is,

$$A = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}} \approx -20\log\sqrt{\frac{\omega^4}{\omega_n^4}} = -20\log\frac{\omega^2}{\omega_n^2} = -20\log\left(\frac{\omega}{\omega_n}\right)^2$$

$$\therefore A = -40\log\frac{\omega}{\omega_n}$$

At $\omega = \omega_n$, $A = -40\log 1 = 0$ db

At $\omega = 10\omega_n$, $A = -40\log 10 = -40$ db

From the above analysis it is evident that the magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope –40 db/dec for the frequency range $\omega_n < \omega < \infty$. The two straight lines are asymptotes of the exact curve. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency ω_n is the corner frequency, ω_c.

The two asymptotes of the exact curve are independent of the damping ratio, ζ. In the exact magnitude plot, resonant peak occurs near the corner frequency and the magnitude of resonant peak depends on ζ. Lower the value of ζ, larger will be the resonant peak. Hence by this approximation the error at the corner frequency depends on damping ratio ζ. The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω.

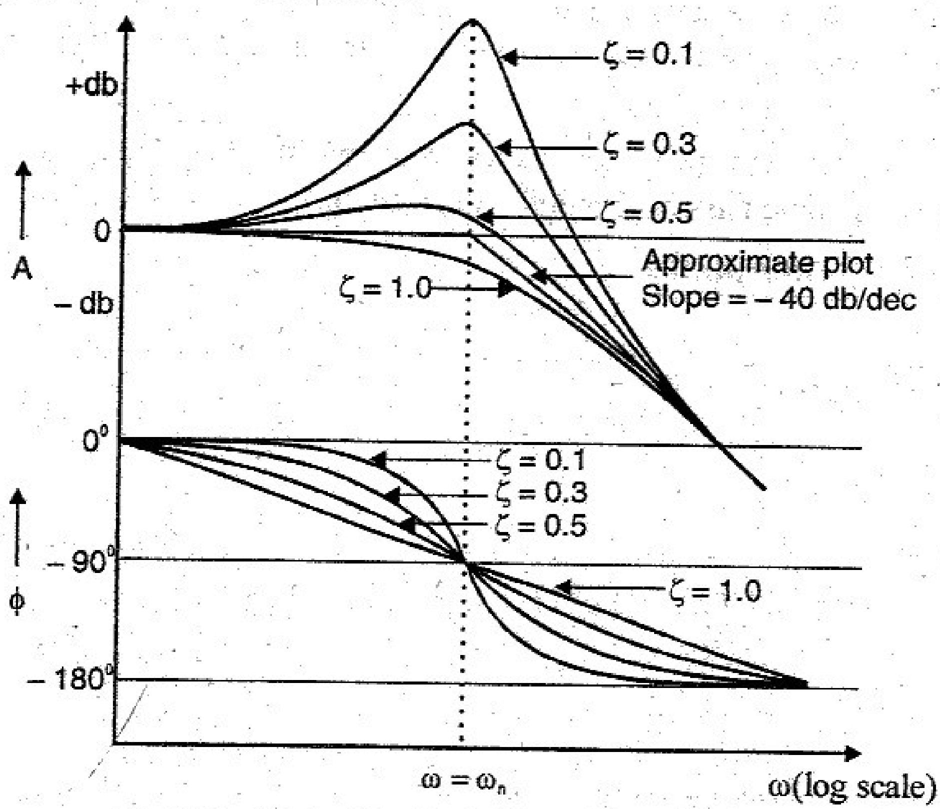


Fig 3.16 : Bode plot of quadratic factor in denominator.

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

As $\omega = \omega_n$, $\phi = -\tan^{-1}\frac{2\zeta}{0} = -\tan^{-1}\infty = -90°$

As $\omega \to 0$, $\phi \to 0$

As $\omega \to \infty$, $\phi \to -180°$

The phase angle of the quadratic factor varies from 0 to -180^0 as ω is varied from 0 to ∞. The phase plot is a curve passing through -90^0 at ω_c. At the corner frequency phase angle is -90^0 and independent of ζ, but at all other frequency it depends on ζ.

QUADRATIC FACTOR IN THE NUMERATOR

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = 1 + 2\zeta\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$G(j\omega) = 1 + j2\zeta\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}} \angle\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

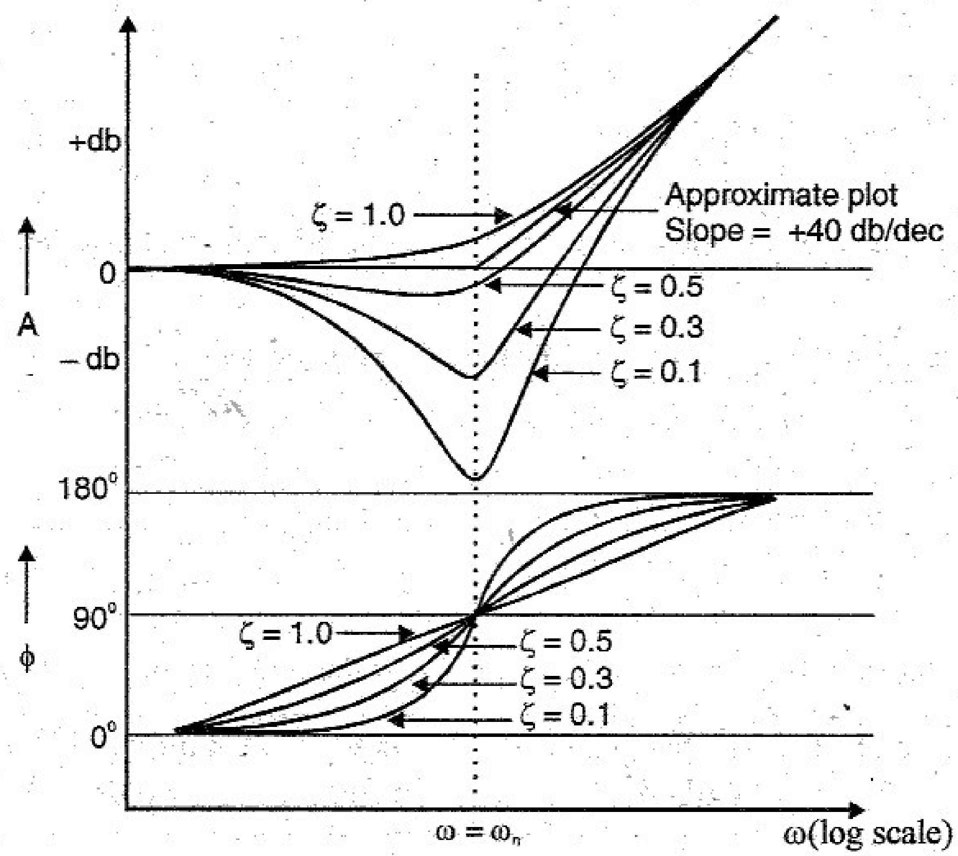


Fig 3.17 : Bode plot of quadratic factor in numerator.

Based on an analysis similar to that of denominator quadratic factor, the magnitude plot of the quadratic factor in the numerator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope +40 db/dec for the frequency range $\omega_n < \omega < \infty$. The corner frequency is ω_n. Due to this approximation the error at the corner frequency depends on ζ.

The phase angle varies from 0 to +180°, as ω is varied from 0 to ∞. At the corner frequency the phase angle is +90° and independent of ζ, but at all other frequency it depends on ζ.

PROCEDURE FOR MAGNITUDE PLOT OF BODE PLOT

From the analysis of previous sections the following conclusions can be obtained.

1. The constant gain K, integral and derivative factors contribute gain (magnitude) at all frequencies.
2. In approximate plot the first, quadratic and higher order factors contribute gain (magnitude) only when the frequency is greater than the corner frequency.

Hence the low frequency response upto the lowest corner frequency is decided by K or $K/(j\omega)^n$ or $K(j\omega)^n$ term. Then at every corner frequency the slope of the magnitude plot is altered by the first, quadratic and higher order terms. Therefore the magnitude plot can be started with K or $K/(j\omega)^n$ or $K(j\omega)^n$ term and then the db magnitude of every first and higher order terms are added one by one in the increasing order of the corner frequency.

This is illustrated in the following example.

Let, $G(s) = \dfrac{K(1+sT_1)^2}{s^2(1+sT_2)(1+sT_3)}$

$$\therefore G(j\omega) = \frac{K(1+j\omega T_1)^2}{(j\omega)^2(1+j\omega T_2)(1+j\omega T_3)}$$

Let, $T_2 < T_3 < T_1$.

The corner frequencies are, $\omega_{c1} = \dfrac{1}{T_1}$, $\omega_{c2} = \dfrac{1}{T_2}$, $\omega_{c3} = \dfrac{1}{T_3}$.

Let, $\omega_{c1} < \omega_{c3} < \omega_{c2}$.

The magnitude plot of the individual terms of $G(j\omega)$, and their combined magnitude plot are shown in fig 3.18.

The step by step procedure for plotting the magnitude plot is given below

Step1 : Convert the transfer function into Bode form or time constant form. The Bode form of the transfer function is

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)\left(1+\dfrac{s^2}{\omega_n^2}+2\zeta\dfrac{s}{\omega_n}\right)} \xrightarrow{s=j\omega} G(j\omega) = \frac{K(1+j\omega T_1)}{j\omega(1+j\omega T_2)\left(1-\dfrac{\omega^2}{\omega_n^2}+j2\zeta\dfrac{\omega}{\omega_n}\right)}$$

Step 2 : List the corner frequencies in the increasing order and prepare a table as shown below.

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec

In the above table enter K or $K/(j\omega)^n$ or $K(j\omega)^n$ as the first term and the other terms in the increasing order of corner frequencies. Then enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

Step 3 : Choose an arbitrary frequency ω_l which is lesser than the lowest corner frequency. Calculate the db magnitude of K or $K/(j\omega)^n$ or $K(j\omega)^n$ at ω_l and at the lowest corner frequency.

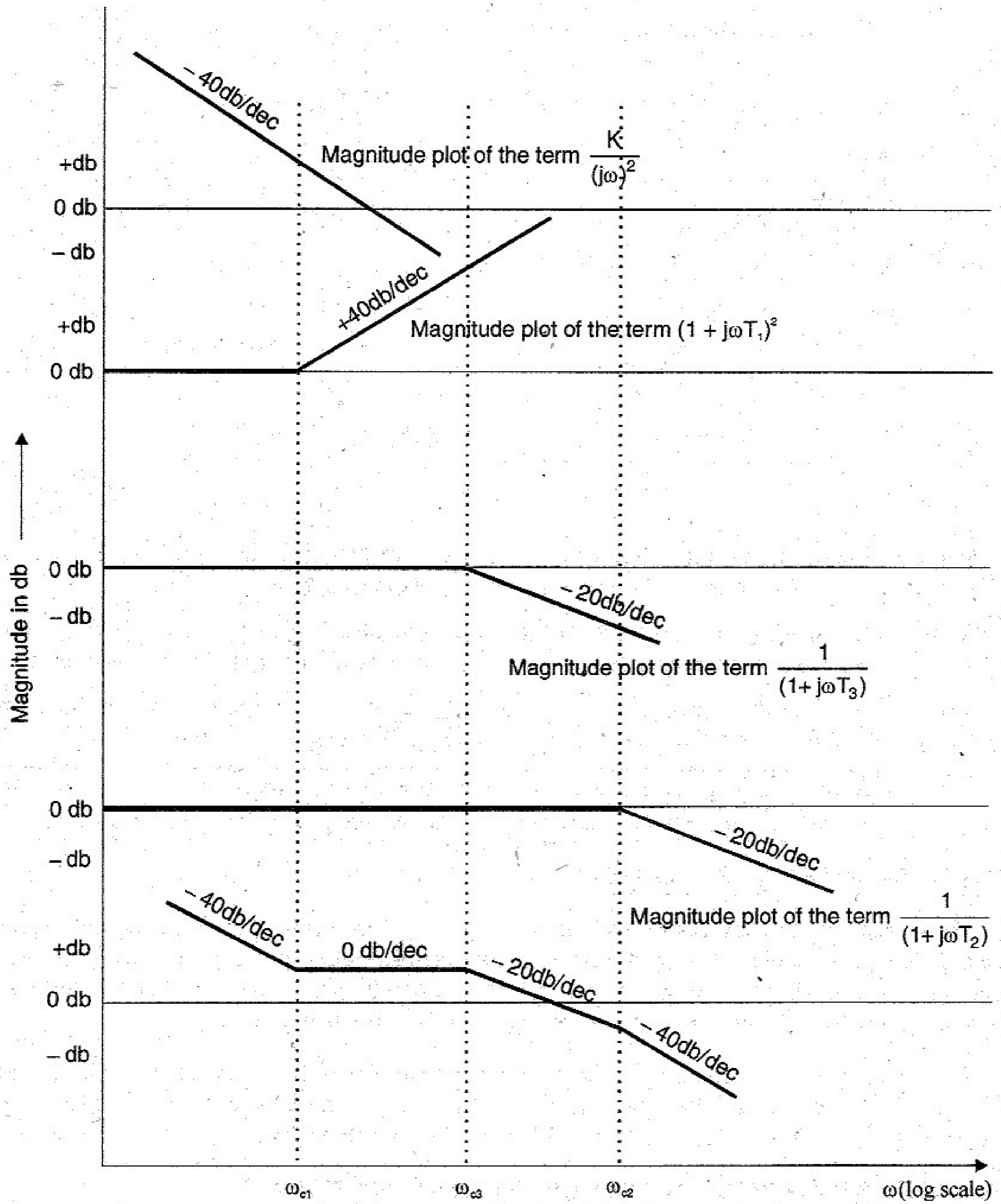
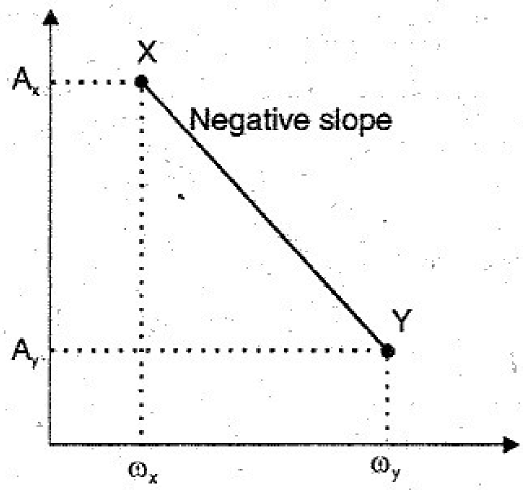


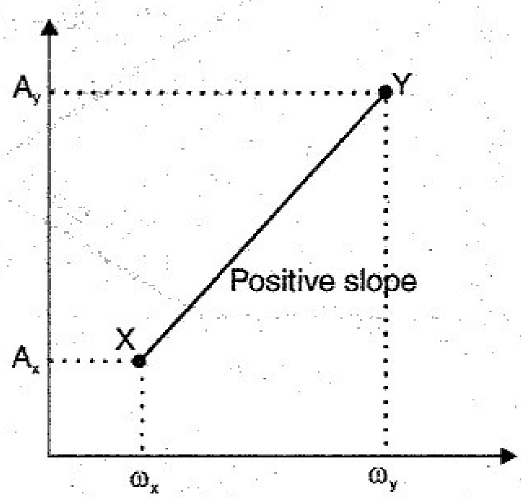
Fig 3.18 : *Magnitude plot of bode plot of,* $G(j\omega) = \dfrac{K(1+ j\omega T_1)^2}{(j\omega)^2(1+ j\omega T_2)(1+ j\omega T_3)}$.

Step 4 : Then calculate the gain (db magnitude) at every corner frequency one by one by using the formula,

Gain at ω_y = change in gain from ω_x to ω_y + Gain at ω_x

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log\frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$





Step 5 : Choose an arbitrary frequency ω_h which is greater than the highest corner frequency. Calculate the gain at ω_h by using the formula in step 4.

Step 6 : In a semilog graph sheet mark the required range of frequency on x-axis (log scale) and the range of db magnitude on y-axis (ordinary scale) after choosing proper units.

Step 7 : Mark all the points obtained in steps 3, 4, and 5 on the graph and join the points by straight lines. Mark the slope at every part of the graph.

> ***Note :*** *The magnitude plot obtained above is an approximate plot. If an exact plot is needed then appropriate corrections should be made at every corner frequencies.*

PROCEDURE FOR PHASE PLOT OF BODE PLOT

The phase plot is an exact plot and no approximations are made while drawing the phase plot. Hence the exact phase angles of G(jω) are computed for various values of ω and tabulated. The choice of frequencies are preferably the frequencies chosen for magnitude plot. Usually the magnitude plot and phase plot are drawn in a single semilog - sheet on a common frequency scale.

Take another y-axis in the graph where the magnitude plot is drawn and in this y-axis mark the desired range of phase angles after choosing proper units. From the tabulated values of ω and phase angles, mark all the points on the graph. Join the points by a smooth curve.

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM BODE PLOT

The gain margin in db is given by the negative of db magnitude of G(jω) at the phase cross-over frequency, ω_{pc}. The ω_{pc} is the frequency at which phase of G(jω) is $-180°$. If the db magnitude of G(jω) at ω_{pc} is negative then gain margin is positive and vice versa.

Let ϕ_{gc} be the phase angle of G(jω) at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of G(jω) is zero. Now the phase margin, γ is given by, $\gamma = 180° + \phi_{gc}$. If ϕ_{gc} is less negative than $-180°$ then phase margin is positive and vice versa.

The positive and negative gain margins and phase margins are illustrated in fig 3.19.

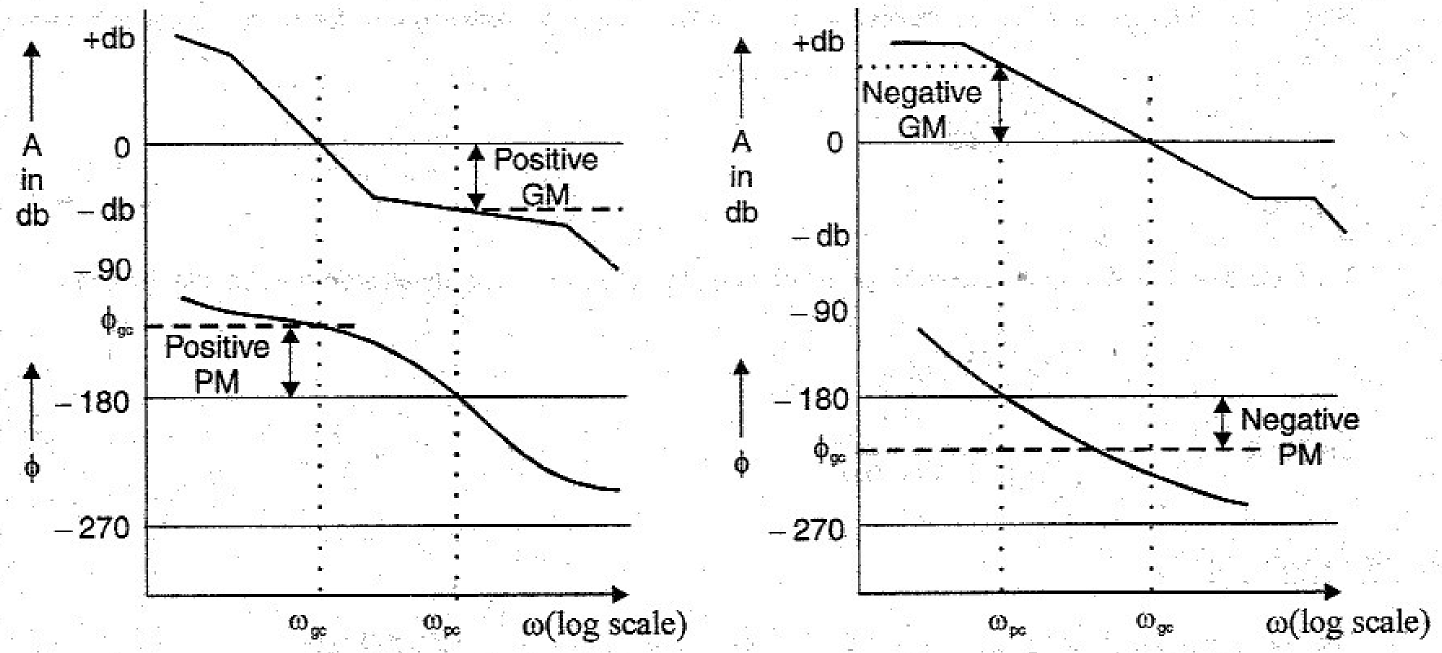


Fig 3.19 : Bode plot showing phase margin (PM) and gain margin (GM).

GAIN ADJUSTMENT IN BODE PLOT

In the open loop transfer function G(jω) the constant K contributes only magnitude. Hence by changing the value of K the system gain can be adjusted to meet the desired specifications. The desired specifications are gain margin, phase margin, ω_{pc} and ω_{gc}. In a system transfer function if the value of K required to be estimated to satisfy a desired specification then draw the bode plot of the system with K = 1. The constant K can add 20 log K to every point of the magnitude plot and due to this addition the magnitude plot will shift vertically up or down. Hence shift the magnitude plot vertically up or down to meet the desired specification. Equate the vertical distance by which the magnitude plot is shifted to 20 log K and solve for K.

Let, x = change in db   (x is positive if the plot is shifted up and vice versa).

Now, $20 \log K = x$; $\log K = x/20$; $\therefore\ K = 10^{x/20}$

> ***Note :*** *A point in complex plane can be represented by rectangular coordinates or by polar coordinates. Consider a point, z = a+jb in complex plane.*
>
> *Now,* $|z| = \sqrt{a^2 + b^2}$ *and* $\angle z = \tan^{-1} b/a$.
>
> *If the point lies in first or fourth quadrant then the argument as calculated by* $\tan^{-1} b/a$ *will be the correct values. But if it lies either in second or third quadrant then a correction should be made in the calculated values of argument, because the calculator will always give the values of* $\tan^{-1} b/a$ *either from 0 to + 90° or from 0 to–90°. The corrections to be made while converting from rectangular to polar coordinates is shown below.*
>
> A point in I^{st} quadrant, $a + jb = \sqrt{a^2 + b^2}\ \angle \tan^{-1} b/a$
>
> A point in II^{nd} quadrant, $-a + jb = \sqrt{a^2 + b^2}\ \angle(\pi - \tan^{-1} b/a)$
>
> A point in III^{rd} quadrant, $-a - jb = \sqrt{a^2 + b^2}\ \angle(\pi + \tan^{-1} b/a)$
>
> A point in IV^{th} quadrant, $a - jb = \sqrt{a^2 + b^2}\ \angle - \tan^{-1} b/a)$

EXAMPLE 3.1

Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec.

$$G(s) = \frac{Ks^2}{(1+0.2s)\,(1+0.02s)}$$

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in the given s-domain transfer function.

$$\therefore\; G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

$$\text{Let } K=1, \quad \therefore\; G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$ and $\omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}$

The various terms of G(jω) are listed in Table-1 in the increasing order of their corner frequency. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$(j\omega)^2$	–	+40	
$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = \frac{1}{0.2} = 5$	–20	$40-20=20$
$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = \frac{1}{0.02} = 50$	20	$20-20=0$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.5$ rad/sec and $\omega_h = 100$ rad/sec.

Let, $A = |G(j\omega)|$ in db.

Let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h.

$$\text{At } \omega=\omega_l, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(0.5)^2 = -12 \text{ db}$$

$$\text{At } \omega=\omega_{c1}, \quad A = 20\log\left|(j\omega)^2\right| = 20\log(\omega)^2 = 20\log(5)^2 = 28 \text{ db}$$

$$\text{At } \omega=\omega_{c2}, \quad A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})} = 20\times\log\frac{50}{5} + 28 = 48 \text{ db}$$

$$\text{At } \omega=\omega_h, \quad A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})} = 0\times\log\frac{100}{50} + 48 = 48 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on, logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.02\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg	Point in phase plot
0.5	5.7	0.6	$173.7 \approx 174$	e
1	11.3	1.1	$167.6 \approx 168$	f
5	45	5.7	$129.3 \approx 130$	g
10	63.4	11.3	$105.3 \approx 106$	h
50	84.3	45	$50.7 \approx 50$	i
100	87.1	63.4	$29.5 \approx 30$	j

On the same semilog graph sheet choose a scale of 1unit = 20°, on the y-axis on the right side of semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

CALCULATION OF K

Given that the gain crossover frequency is 5 rad/sec. At ω = 5 rad/sec the gain is 28 db. If gain crossover frequency is 5 rad/sec then at that frequency the db gain should be zero. Hence to every point of magnitude plot a db gain of –28db should be added. The addition of –28db shifts the plot downwards. The corrected magnitude plot is obtained by shifting the plot with K = 1 by –28db downwards. The magnitude correction is independent of frequency. Hence the magnitude of –28db is contributed by the term K. The value of K is calculated by equating 20 logK to –28 db.

$$\therefore 20 \log K = -28 \text{ db}$$

$$\log K = \frac{-28}{20}\ ; \quad K = 10^{-\left(\frac{28}{20}\right)} = 0.0398$$

The magnitude plot with K = 1 and 0.0398 and the phase plot are shown in fig 3.1.1

Note : The frequency ω = 5 rad/sec is a corner frequency. Hence in the exact plot the db gain at ω = 5 rad/sec will be 3db less than the actual plot. Therefore for exact plot the 20 log K will contribute a gain of –25db.

$$\therefore 20 \log K = -25 \text{ db}$$

$$\log K = \frac{-25}{20}\ ; \quad K = 10^{-\left(\frac{25}{20}\right)} = 0.0562$$

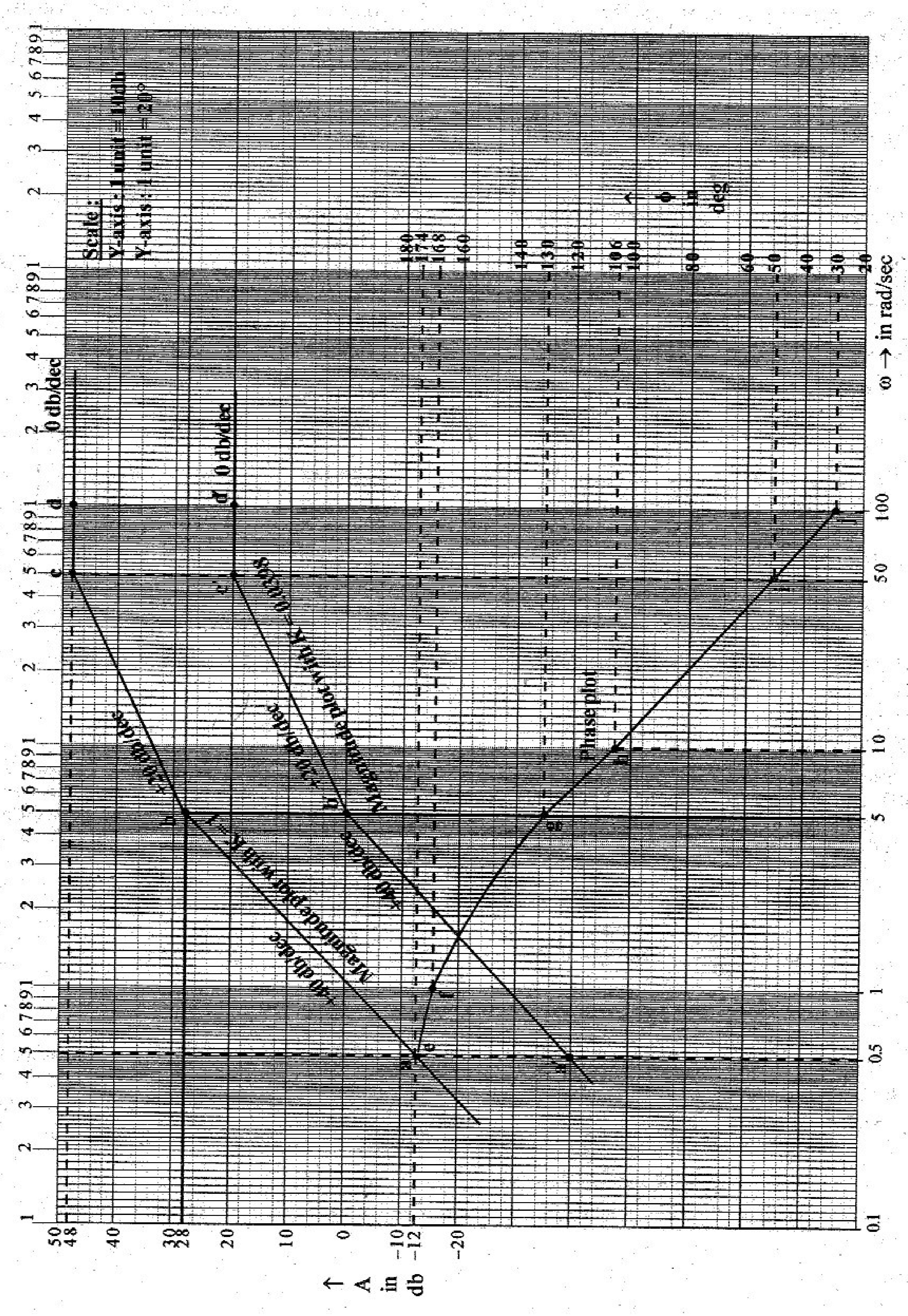


Fig 3.1.1 : Bode plot of transfer function, $G(j\omega) = \dfrac{K(j\omega)^2}{(1 + j0.2\omega)(1 + j0.02\omega)}$

EXAMPLE 3.2

Sketch the bode plot for the following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75\,(1+0.2s)}{s\left(s^2+16s+100\right)}$$

SOLUTION

On comparing the quadratic factor in the denominator of G(s) with standard form of quadratic factor we can estimate ζ and ω_n.

$$\therefore s^2+16s+100 = s^2+2\zeta\omega_n s+\omega_n^2$$

On comparing we get,

$$\omega_n^2 = 100 \quad \Rightarrow \quad \omega_n = 10$$

$$2\zeta\omega_n = 16 \quad \Rightarrow \quad \zeta = \frac{16}{2\omega_n} = \frac{16}{2\times 10} = 0.8$$

Let us convert the given s-domain transfer function into bode form or time constant form.

$$\therefore G(s) = \frac{75(1+0.2s)}{s\left(s^2+16s+100\right)} = \frac{75\,(1+0.2s)}{s\times 100\left(\frac{s^2}{100}+\frac{16s}{100}+1\right)} = \frac{0.75\,(1+0.2s)}{s\left(1+0.01s^2+0.16s\right)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in G(s).

$$\therefore G(j\omega) = \frac{0.75\,(1+0.2\,j\omega)}{j\omega\left(1+0.01(j\omega)^2+0.16j\omega\right)} = \frac{0.75\,(1+j0.2\omega)}{j\omega\left(1-0.01\omega^2+j0.16\omega\right)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$ and $\omega_{c2} = \omega_n = 10 \text{ rad/sec}$

Note : For the quadratic factor the corner frequency is ω_n.

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{0.75}{j\omega}$	–	–20	
$1+j0.2\omega$	$\omega_{c1} = \frac{1}{0.2} = 5$	20	$-20+20=0$
$\frac{1}{1-0.01\omega^2+j0.16\omega}$	$\omega_{c2} = \omega_n = 10$	– 40	$0-40=-40$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.5$ rad/sec and $\omega_h = 20$ rad/sec.

Let, $A = |G(j\omega)|$ in db .

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At,} \quad \omega = \omega_l, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{0.5} = 3.5\text{ db}$$

$$\text{At,} \quad \omega = \omega_{c1}, \quad A = 20\log\left|\frac{0.75}{j\omega}\right| = 20\log\frac{0.75}{5} = -16.5\text{ db}$$

$$\text{At,} \quad \omega = \omega_{c2}, \; A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})}$$

$$= 0 \times \log\frac{10}{5} + (-16.5) = -16.5\text{ db}$$

$$\text{At } \omega = \omega_h, \; A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$$

$$= -40 \times \log\frac{20}{10} + (-16.5) = -28.5\text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a,b,c and d on the graph. Join the points by straight lines and mark the slope on the respective region.

Note : In quadratic factors the phase varies from 0° to 180°. But calculator calculates $\tan^{-1}$ only between 0° to 90°. Hence a correction of 180° should be added to phase after ω_n.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} \quad \text{for } \omega \le \omega_n$$

$$\phi = \angle G(j\omega) = \tan^{-1}0.2\omega - 90^\circ - \left(\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2} + 180^\circ\right) \quad \text{for } \omega > \omega_n$$

The phase angle of G(jω) are calculated for various values of w and listed in Table-2.

TABLE-2

ω rad/sec	$\tan^{-1}0.2\,\omega$ deg	$\tan^{-1}\frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.5	5.7	4.6	$-88.9 \approx -88$	e
1	11.3	9.2	$-87.9 \approx -88$	f
5	45	46.8	$-91.8 \approx -92$	g
10	63.4	90	$-116.6 \approx -116$	h
20	75.9	$-46.8+180=133.2$	$-147.3 \approx -148$	i
50	84.3	$-18.4+180=161.6$	$-167.3 \approx -168$	j
100	87.1	$-92+180=170.8$	$-173.7 \approx -174$	k

On the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude plot and the phase plot are shown in fig 3.2.1.

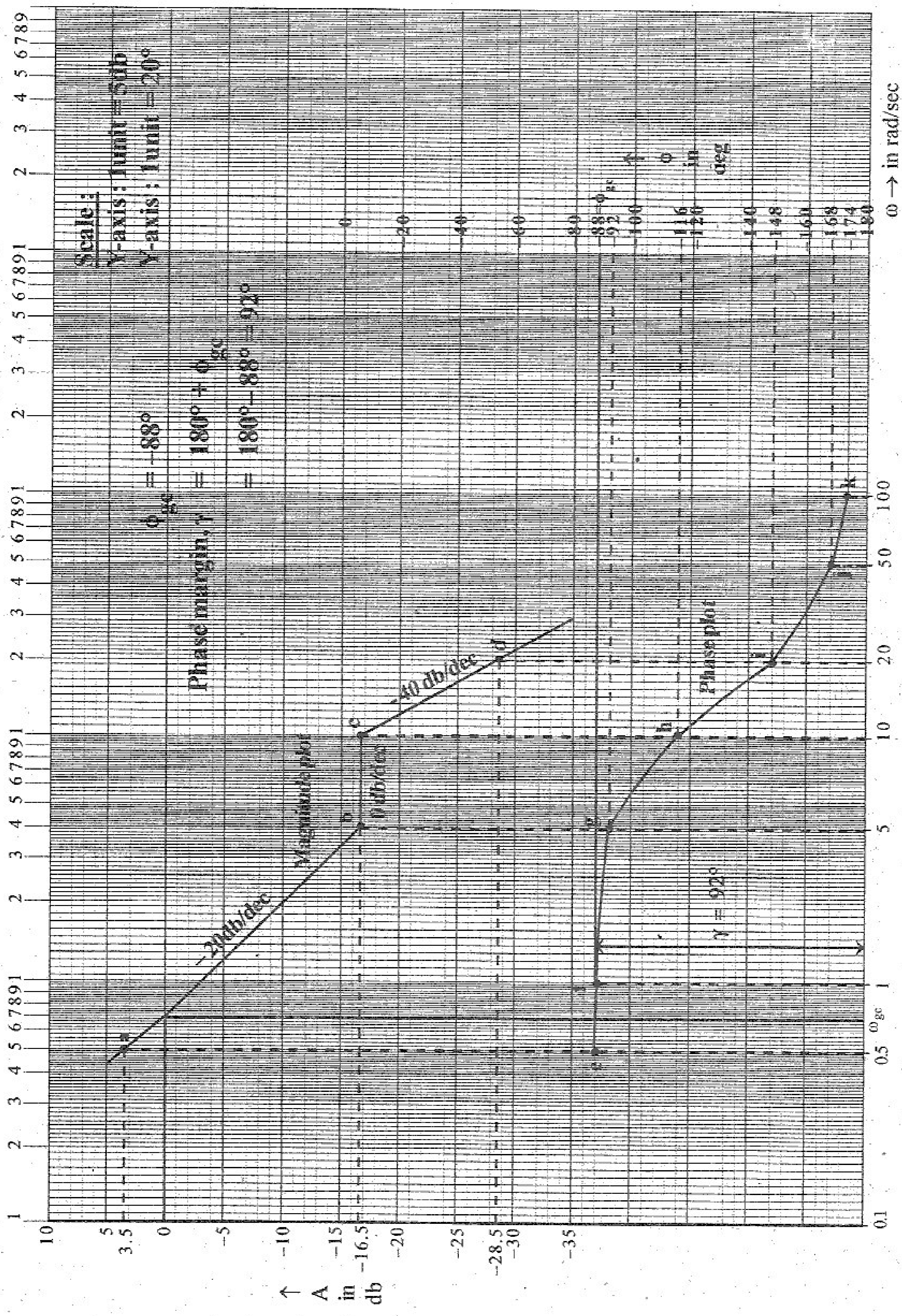


Fig 3.2.1 : Bode plot of transfer function, $G(j\omega) = \dfrac{0.75(1 + j0.2\omega)}{j\omega(1 - 0.01\omega^2 + j0.16\omega)}$

Let ϕ_{gc} be the phase of $G(j\omega)$ at gain cross-over frequency, ω_{gc}.

From the fig 3.2.1, we get, $\phi_{gc} = 88°$

$$\therefore \text{Phase margin, } g = 180° + \phi_{gc} = 180° - 88° = 92°$$

The phase plot crosses $-180°$ only at infinity. The $|G(j\omega)|$ at infinity is $-\infty$ db.

Hence gain margin is $+\infty$.

EXAMPLE 3.3

Given, $G(s) = \dfrac{K\,e^{-0.2s}}{s\,(s+2)\,(s+8)}$. Find K so that the system is stable with,

(a) gain margin equal to 2db, (b) phase margin equal to 45°.

SOLUTION

Let us take K = 1, and convert the given transfer function to time constant form or bode form.

$$\therefore G(s) = \frac{e^{-0.2s}}{s\,(s+2)(s+8)} = \frac{e^{-0.2s}}{s \times 2\left(1+\dfrac{s}{2}\right) \times 8\left(1+\dfrac{s}{8}\right)} = \frac{0.0625\;e^{-0.2s}}{s\,(1+0.5s)(1+0.125s)}$$

The sinusoidal transfer function $G(j\omega)$ is obtained by replacing s by $j\omega$ in G(s).

$$\therefore G(j\omega) = \frac{0.0625\;e^{-j0.2\omega}}{j\omega\;(1+j0.5\omega)\;(1+j0.125\omega)}$$

Note : $\left|0.0625\,e^{-j0.2\omega}\right| = 0.0625$ *and* $\angle(0.0625e^{-j0.2\omega}) = -0.2\omega$ *radians.*

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \dfrac{1}{0.5} = 2\text{ rad/sec}$ and $\omega_{c2} = \dfrac{1}{0.125} = 8\text{ rad/sec}$

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\dfrac{0.0625}{j\omega}$	—	-20	
$\dfrac{1}{1+j0.5\omega}$	$\omega_{c1} = \dfrac{1}{0.5} = 2$	-20	$-20-20=-40$
$\dfrac{1}{1+j0.125\omega}$	$\omega_{c2} = \dfrac{1}{0.125} = 8$	-20	$-40-20=-60$

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.5$ rad / sec and $\omega_h = 50$ rad / sec.

Let, $A = |G(j\omega)|$ in db.

Let us calculate A at $\omega_l, \omega_{c1}, \omega_{c2}$ and ω_h.

$$\text{At } \omega = \omega_l,\ A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{0.5} = -18\text{db}$$

$$\text{At } \omega = \omega_{c1},\ A = 20\log\left|\frac{0.0625}{j\omega}\right| = 20\log\frac{0.0625}{2} = -30\text{db}$$

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})} = -40\times\log\frac{8}{2} + (-30) = -54\text{db}$$

$$\text{At } \omega = \omega_h,\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})} = -60\times\log\frac{50}{8} + (-54) = -102\text{db}$$

Let the points a,b,c,d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10db on y-axis. The frequencies are marked in decades from 0.01 to 100 rad/sec on logarithmic scale in x-axis. Fix the points a, b, c, and d on the graph. Join the points by straight line and mark the slope on the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = -0.2\omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.125\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	−0.2 ω (180°/π) deg	$\tan^{-1}$ 0.5 ω deg	$\tan^{-1}$ 0.125 ω deg	φ = ∠G(jω) deg	Point in phase plot
0.01	−0.1145	0.2864	0.0716	−90.4 ≈ −90	e
0.1	−1.145	2.862	0.716	−94.7 ≈ −94	f
0.5	−5.7	14	3.6	−113.3 ≈ −114	g
1	−11.4	26	7.12	−134.4 ≈ −134	h
2	−22.9	45	14	−171.9 ≈ −172	i
3	−34.37	56.30	20.56	−201.2 ≈ −202	j
4	−45.84	63.43	26.57	−225.8 ≈ −226	k

On the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of the semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by smooth curve.

The magnitude and phase plot are shown in fig 3.3.1.

CALCULATION OF K

Phase margin, $\gamma = 180^0 + \phi_{gc}$, where ϕ_{gc} is the phase of G(jω) at $\omega = \omega_{gc}$.

When $\gamma = 45°$, $\phi_{gc} = \gamma - 180^0 = 45^0 - 180^0 = -135°$.

With K = 1, the db gain at φ = −135° is −24 db. This gain should be made zero to have to PM of 45°. Hence to every point of magnitude plot a db gain of 24 db should be added. The corrected magnitude plot is obtained by shifting the plot with K = 1 by 24 db upwards. The magnitude correction is independent of frequency. Hence the magnitude of 24 db is contributed by the term K. The value of K is calculated by equating 20logK to 24 db.

$$\therefore 20\log K = 24 \quad ; \quad K = 10^{24/20} \quad ; \quad K = 15.84$$

With K = 1, the gain margin = −(−32) = 32 db. But the required gain margin is 2 db. Hence to every point of magnitude plot a db gain of 30 db should be added. This addition of 30 db shifts the plot upwards. The magnitude correction is independent of frequency. Hence the magnitude of 30 db is contributed by the term K. The value of K is calculated by equating 20logK to 30 db.

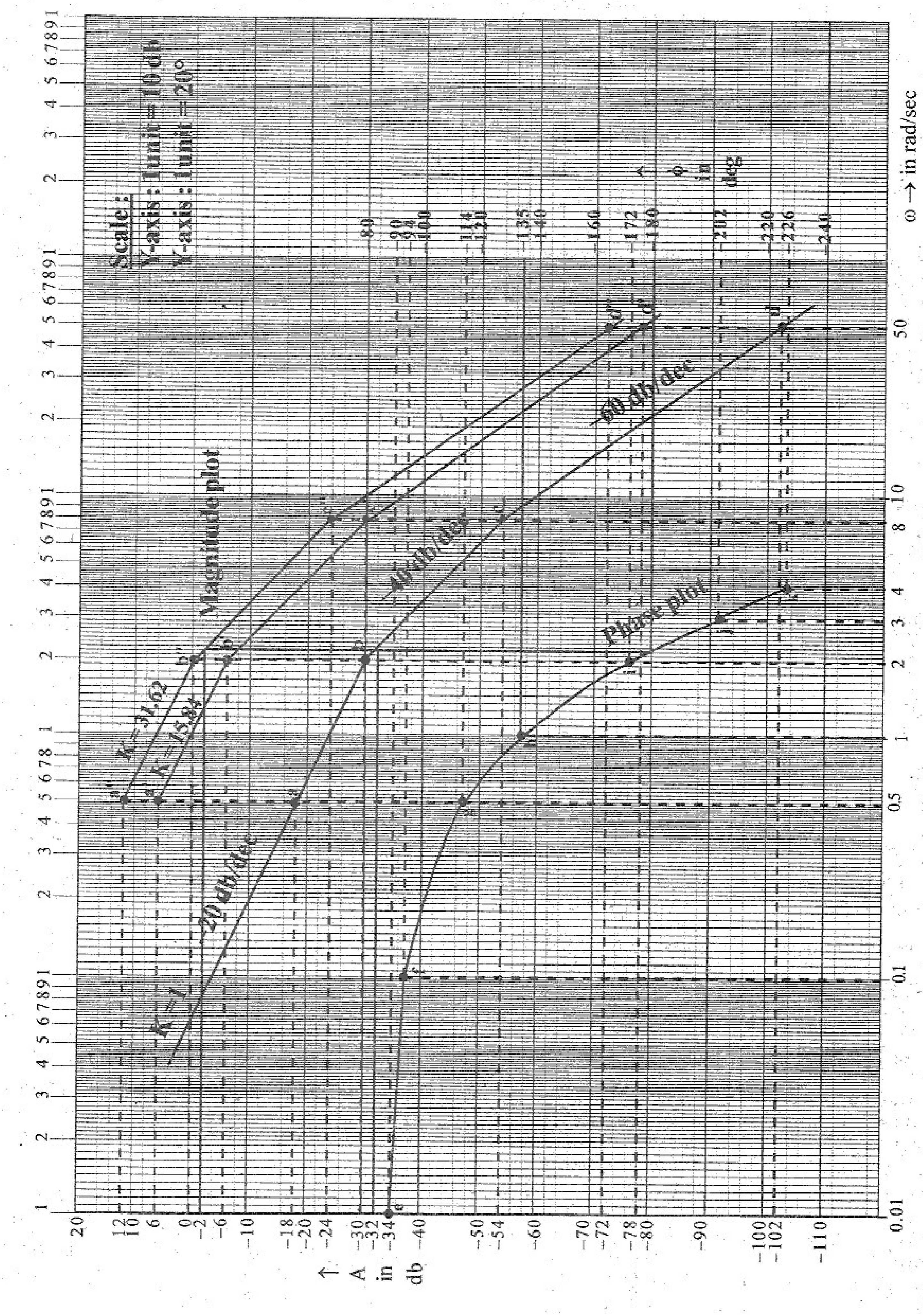


Fig 3.3.1 : *Bode plot of transfer function,* $G(j\omega) = \dfrac{0.0625Ke^{-j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$

$\therefore 20\log K = 30$; $K = 10^{30/20}$; $K = 31.62$

The magnitude plot with K = 15.84 and 31.62 are shown in fig 3.3.1.

EXAMPLE 3.4

Plot the Bode diagram for the following transfer function and obtain the gain and phase cross over frequencies.

$$G(s) = \frac{10}{s\,(1+0.4s)\,(1+0.1s)}$$

SOLUTION

The sinusoidal transfer function of $G(j\omega)$ is obtained by replacing s by $j\omega$ in the given transfer function.

$$\therefore G(j\omega) = \frac{10}{j\omega\,(1+j0.4\omega)\,(1+j0.1\omega)}$$

MAGNITUDE PLOT

The corner frequencies are,

$$\omega_{c1} = \frac{1}{0.4} = 2.5\text{ rad/sec} \quad \text{and} \quad \omega_{c2} = \frac{1}{0.1} = 10\text{ rad/sec}$$

The various terms of $G(j\omega)$ are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by each term and the change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{10}{j\omega}$	–	– 20	
$\frac{1}{1+j0.4\omega}$	$\omega_{c1} = \frac{1}{0.4} = 2.5$	– 20	–20 –20 = – 40
$\frac{1}{1+j0.1\omega}$	$\omega_{c2} = \frac{1}{0.1} = 10$	– 20	– 40 – 20 = – 60

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.1$ rad/sec, and $\omega_h = 50$ rad/sec.

Let, $A = |G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At } \omega = \omega_l,\quad A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{0.1} = 40\text{ db}$$

$$\text{At } \omega = \omega_{c1},\ A = 20\log\left|\frac{10}{j\omega}\right| = 20\log\frac{10}{2.5} = 12\text{ db}$$

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega = \omega_{c1})} = -40 \times \log\frac{10}{2.5} + 12 = -12\text{ db}$$

$$\text{At } \omega = \omega_h,\quad A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})} = -60 \times \log\frac{50}{10} + (-12) = -54\text{ db}$$

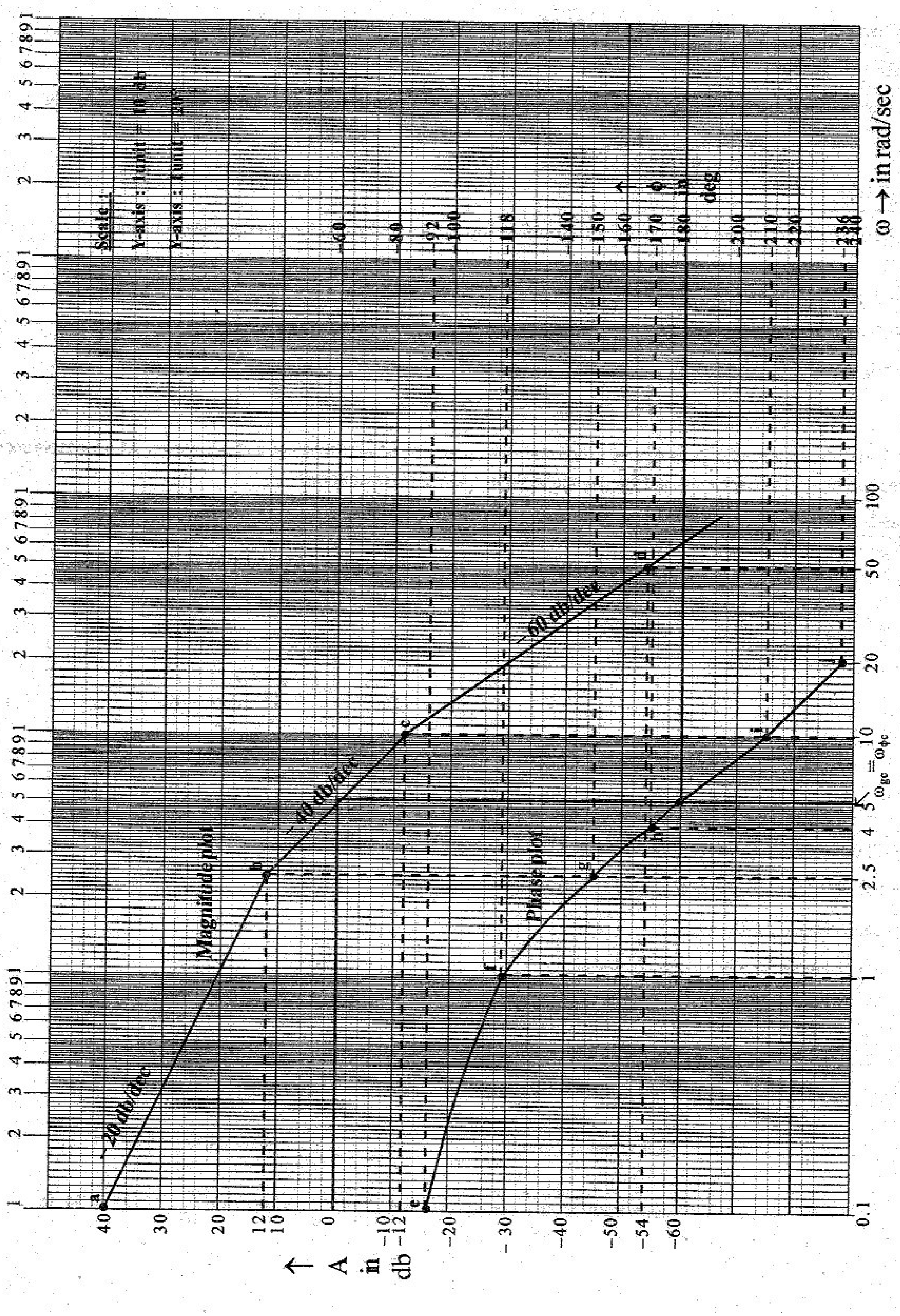


Fig 3.4.1 : Bode plot of transfer function, $G(j\omega) = \dfrac{10}{j\omega(1+0.4\omega)(1+j0.1\omega)}$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 10 db on y-axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales in x-axis. Fix the points a, b, c and d on the graph. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

The phase angle of G(jω) as a function of ω is given by,

$$\phi = -90° - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$$

The phase angle of G(jω) are calculated for various values of ω and listed in table-2.

TABLE-2

ω rad/sec	$\tan^{-1} 0.4\omega$ deg	$\tan^{-1} 0.1\omega$ deg	$\phi = \angle G(j\omega)$ deg	Points in phase plot
0.1	2.29	0.57	−92.86 ≈ −92	e
1	21.80	5.71	−117.5 ≈ −118	f
2.5	45.0	14.0	−149 ≈ −150	g
4	57.99	21.8	−169.79 ≈ −170	h
10	75.96	45.0	−210.96 ≈ −210	i
20	82.87	63.43	−236.3 ≈ 236	j

On the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve.

The magnitude and phase plots are shown in fig 3.4.1.

From the graph, the gain and phase cross over frequencies are found to be 5 rad/sec.

RESULT

Gain cross-over frequency = 5 rad/sec.

Phase cross-over frequency = 5 rad/sec.

EXAMPLE 3.5

For the following transfer function draw bode plot and obtain gain cross-over frequency.

$$G(s) = \frac{20}{s\,(1+3s)\,(1+4s)}$$

SOLUTION

The sinusoidal transfer function of G(jω) is obtained by replacing s by jω in the given transfer function.

$$G(j\omega) = \frac{20}{j\omega\,(1+j3\omega)\,(1+j4\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \frac{1}{4} = 0.25$ rad / sec, $\omega_{c2} = \frac{1}{3} = 0.333$ rad / sec.

The various terms of G(jω) are listed in table-1 in the increasing order of their frequencies. Also the table shows the slope contributed by each term and change in slope at the corner frequency.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/dec
$\frac{20}{j\omega}$	–	-20	
$\frac{1}{1+j4\omega}$	$\omega_{c1}=\frac{1}{4}=0.25$	-20	$-20-20=-40$
$\frac{1}{1+j3\omega}$	$\omega_{c2}=\frac{1}{3}=0.33$	-20	$-40-20=-60$

Choose a frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$.

Let, $\omega_l = 0.15$ rad/sec and $\omega_h = 1$ rad/sec.

Let, $A = |G(j\omega)|$ in db.

Let us calculate A at ω_l, ω_{c1}, ω_{c2} and ω_h.

$$\text{At } \omega=\omega_l,\ A=|G(j\omega)| = 20\log\left|\frac{20}{0.15}\right| = 42.5 \text{ db}$$

$$\text{At } \omega=\omega_{c1},\ A=|G(j\omega)| = 20\log\left|\frac{20}{0.25}\right| = 38 \text{ db}$$

$$\text{At } \omega=\omega_{c2},\ A=\left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at } \omega=\omega_{c1})}$$

$$= -40 \times \log\frac{0.33}{0.25} + 38 = 33 \text{ db}$$

$$\text{At } \omega=\omega_h,\ A=\left[\text{Slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega=\omega_{c2})}$$

$$= -60 \times \log\frac{1}{0.33} + 33 = 4 \text{ db}$$

Let the points a, b, c and d be the points corresponding to frequencies ω_l, ω_{c1}, ω_{c2} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1 unit = 10 db on y-axis. The frequencies are marked in decades from 0.01 to 10 rad/sec on logarithmic scales on x-axis. Fix the points a, b, c and d on the graph sheet. Join the points by a straight line and mark the slope in the respective region.

PHASE PLOT

The phase angle of $G(j\omega)$, $\phi = -90^0 - \tan^{-1}3\omega - \tan^{-1}4\omega$

The phase angle of $G(j\omega)$ are calculated for various values of ω and listed in table-2.

TABLE-2

ω, rad/sec	$\tan^{-1}3\omega$, deg	$\tan^{-1}4\omega$, deg	$\phi = \angle G(j\omega)$, deg	Points in phase plot
0.15	24.22	30.96	$-145.18 \approx -146$	e
0.2	30.96	38.66	$-159.61 \approx -160$	f
0.25	36.86	45.0	$-171.86 \approx -172$	g
0.33	44.7	52.8	$-187.5 \approx -188$	h
0.6	60.14	67.38	$-218.32 \approx -218$	i
1	71.56	75.96	$-237.56 \approx -238$	j

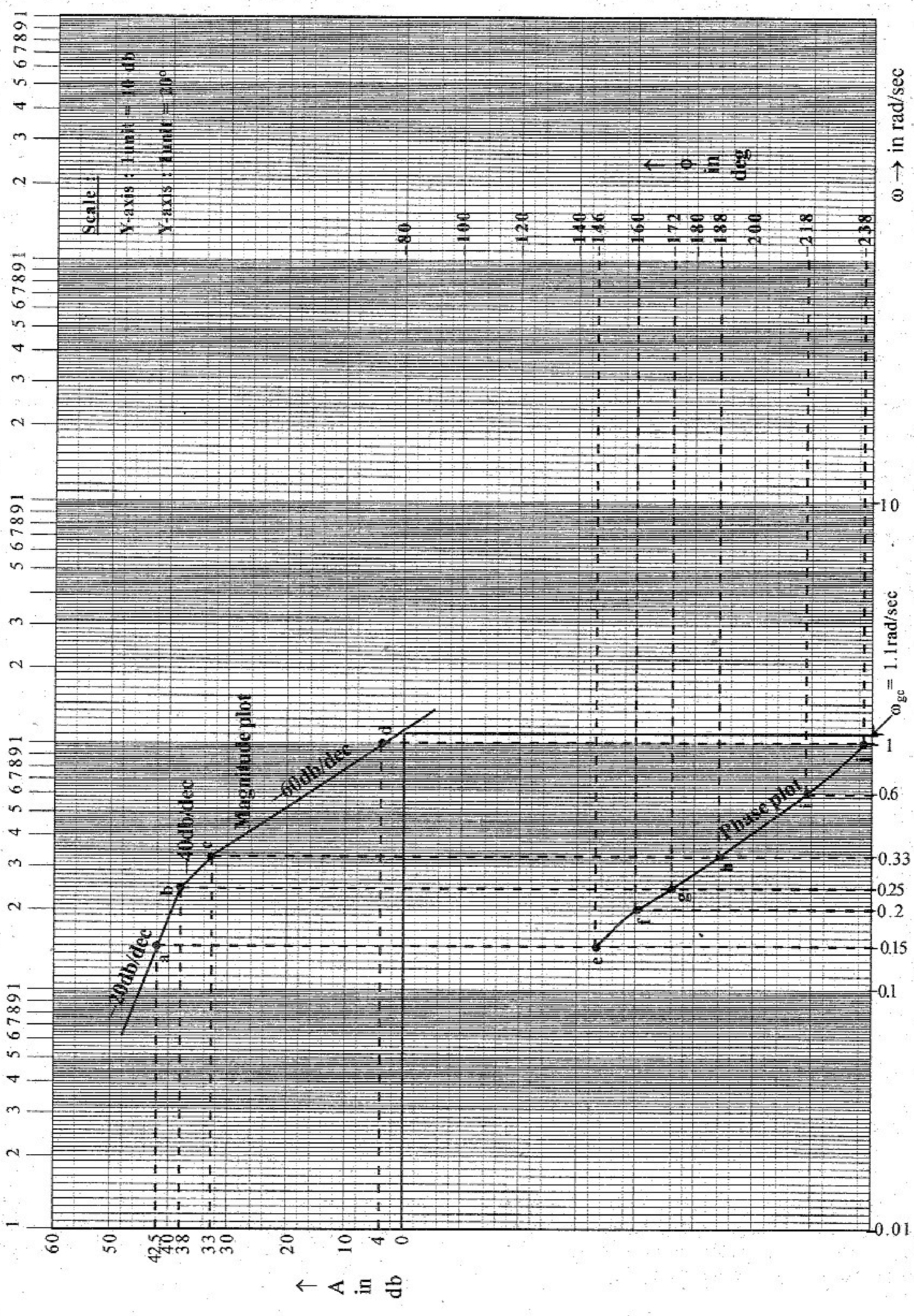


Fig 3.5.1 : *Bode plot for transfer function,* $G(j\omega) = \dfrac{20}{j\omega(1+j3\omega)(1+j4\omega)}$

On the same semilog graph sheet choose a scale of 1unit = 20° on the y-axis on the right side of semilog graph sheet. Mark the calculated phase angle on the graph sheet. Join the points by a smooth curve. The magnitude and phase plots are shown in fig 3.5.1. From the graph the gain cross-over frequency is found to be ω_{gc} = 1.1 rad/sec.

EXAMPLE 3.6

For the function, $G(s) = \dfrac{5(1+2s)}{(1+4s)(1+0.25s)}$, draw the bode plot.

SOLUTION

The sinusoidal transfer function G(jω) is obtained by replacing s by jω in G(s).

$$\therefore G(j\omega) = \frac{5\,(1+j2\omega)}{(1+j4\omega)\,(1+j0.25\omega)}$$

MAGNITUDE PLOT

The corner frequencies are, $\omega_{c1} = \dfrac{1}{4} = 0.25$ rad / sec, $\omega_{c2} = \dfrac{1}{2} = 0.5$ rad / sec, $\omega_{c3} = \dfrac{1}{0.25} = 4$ rad / sec

The various terms of G(jω) are listed in table-1 in the increasing order of their corner frequencies. Also the table shows the slope contributed by the each term and the change in slope at the corner frequency.

Choose a low frequency ω_l such that $\omega_l < \omega_{c1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c2}$. Let ω_l = 0.1 rad/sec and ω_h = 10 rad/sec.

Let A = |G(jω)| in db and let us calculate A at ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h.

TABLE-1

Term	Corner frequency rad/sec	Slope db/dec	Change in slope db/deg
5	–	0	–
$\dfrac{1}{1+j4\omega}$	$\omega_{c1} = \dfrac{1}{4} = 0.25$	–20	0 – 20 = –20
$1+j2\omega$	$\omega_{c2} = \dfrac{1}{2} = 0.5$	20	–20 + 20 = 0
$\dfrac{1}{1+j0.25\omega}$	$\omega_{c3} = \dfrac{1}{0.25} = 4$	–20	0–20 = –20

At $\omega = \omega_l$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

At $\omega = \omega_{c1}$, $A = |G(j\omega)| = 20 \log 5 = +14$ db

$$\text{At } \omega = \omega_{c2},\ A = \left[\text{Slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(\text{at }\omega=\omega_{c1})} = -20 \times \log\frac{0.5}{0.25} + 14 = +8\text{db}$$

$$\text{At } \omega = \omega_{c3},\ A = \left[\text{Slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_{(\text{at }\omega=\omega_{c2})} = 0 \times \log\frac{4}{0.5} + 8 = +8\text{db}$$

$$\text{At } \omega = \omega_{h},\ A = \left[\text{Slope from } \omega_{c3} \text{ to } \omega_{h} \times \log\frac{\omega_{h}}{\omega_{c3}}\right] + A_{(\text{at }\omega=\omega_{c3})} = -20 \log\frac{10}{4} + 8 = 0 \text{ db}$$

Let the points a, b, c, d and e be the points correponding to frequencies ω_l, ω_{c1}, ω_{c2}, ω_{c3} and ω_h respectively on the magnitude plot. In a semilog graph sheet choose a scale of 1unit = 5 db on y axis. The frequencies are marked in decades from 0.1 to 100 rad/sec on logarithmic scales on x-axis. Fix the points a, b, c, d and e on the graph. Join the points by a straight line and mark the slope in the respective region.

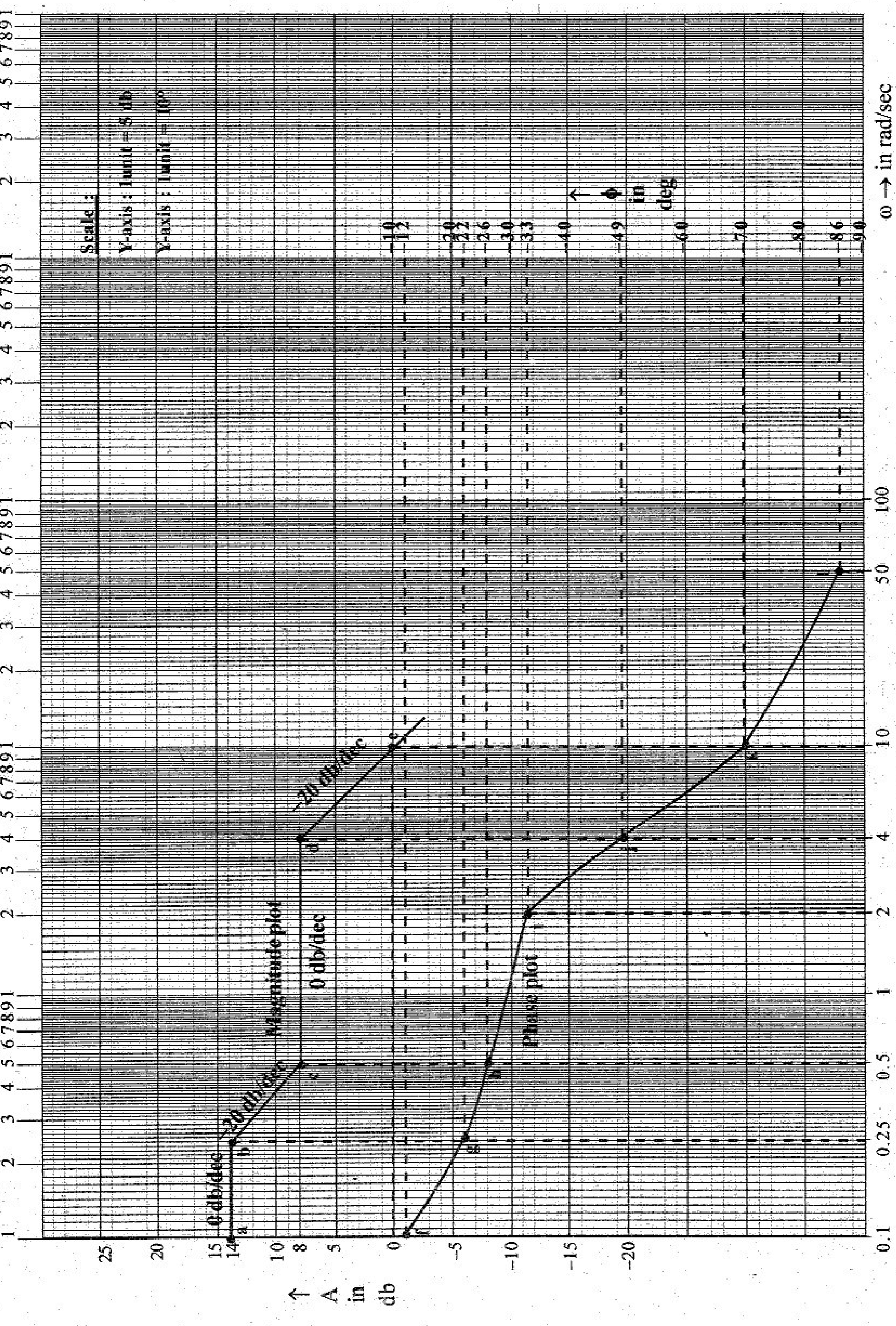


Fig 3.6.1 : *Bode plot of transfer function,* $G(j\omega) = \dfrac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$

PHASE PLOT

The phase angle of G(jω), $\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$

The phase angle of G(jω) are calculated for various values of ω and listed in the table-2.

TABLE-2

ω	$\tan^{-1} 2\omega$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1} 0.25\omega$ deg	$\phi = \angle G(j\omega)$	Points in phase plot
0.1	11.3	21.8	1.43	−11.93 ≈ −12	f
0.25	26.56	45.0	3.5	−21.94 ≈ −22	g
0.5	45.0	63.43	7.1	−25.53 ≈ −26	h
2	75.96	82.87	26.56	−33.47 ≈ −33	i
4	82.87	86.42	45.0	−48.55 ≈ −49	j
10	87.13	88.56	68.19	−69.62 ≈ −70	k
50	89.42	89.71	85.42	−85.71 ≈ −86	l

On the same semilog graph sheet choose a scale of 1 unit = 10° on y-axis on the right side of the semilog graph sheet. Mark the calculated phase angle on the graph sheet, Join the points by a smooth curve. The magnitude and phase plots are shown in fig 3.6.1.

3.7 POLAR PLOT

The polar plot of a sinusoidal transfer function G(jω) is a plot of the magnitude of G(jω) versus the phase angle of G(jω) on polar coordinates as ω is varied from zero to infinity. Thus the polar plot is the locus of vectors |G(jω)| ∠G(jω) as ω is varied from zero to infinity. The polar plot is also called ***Nyquist plot.***

The polar plot is usually plotted on a polar graph sheet. The polar graph sheet has concentric circles and radial lines. The circles represent the magnitude and the radial lines represent the phase angles. Each point on the polar graph has a magnitude and phase angle. The magnitude of a point is given by the value of the circle passing through that point and the phase angle is given by the radial line passing through that point. In polar graph sheet a positive phase angle is measured in anticlockwise from the reference axis (0°) and a negative angle is measured clockwise from the reference axis (0°).

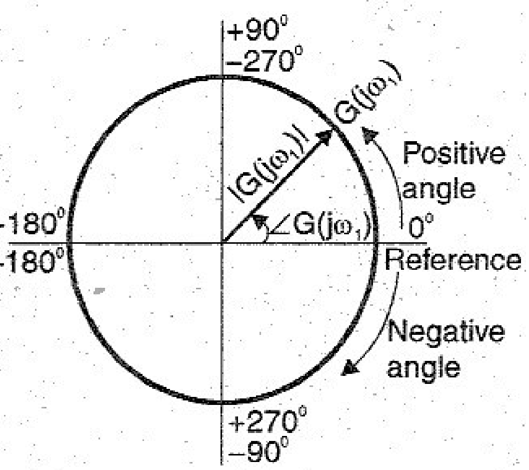


Fig 3.20 : Polar graph.

In order to plot the polar plot, magnitude and phase of G(jω) are computed for various values of ω and tabulated. Usually the choice of frequencies are corner frequencies and frequencies around corner frequencies. Choose proper scale for the magnitude circles. Fix all the points on polar graph sheet and join the points by smooth curve. Write the frequency corresponding to each point of the plot.

Alternatively, if G(jω) can be expressed in rectangular coordinates as,

$$G(j\omega) = G_R(j\omega) + jG_I(j\omega)$$

where, $G_R(j\omega)$ = Real part of G(jω) ; $G_I(j\omega)$ = Imaginary part of G(jω).

then the polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞. In order to plot the polar plot on ordinary graph sheet, the magnitude and phase of G(jω) are computed for various values of ω. Then convert the polar coordinates to rectangular coordinates using **P→R** conversion (polar to rectangular conversion) in the calculator. Sketch the polar plot using rectangular coordinates.

For minimum phase transfer function with only poles, type number of the system determines the quadrant at which the polar plot starts and the order of the system determines the quadrant at which the polar plot ends. The minimum phase systems are systems with all poles and zeros on left half of s-plane. The start and end of polar plot of all pole minimum phase system are shown in fig 3.21 & 3.22 respectively. Some typical sketches of polar plot are shown in table-3.1.

The change in shape of polar plot can be predicted due to addition of a pole or zero.

1. When a pole is added to a system, the polar plot end point will shift by -90^0.
2. When a zero is added to a system the polar plot end point will shift by $+90^0$.

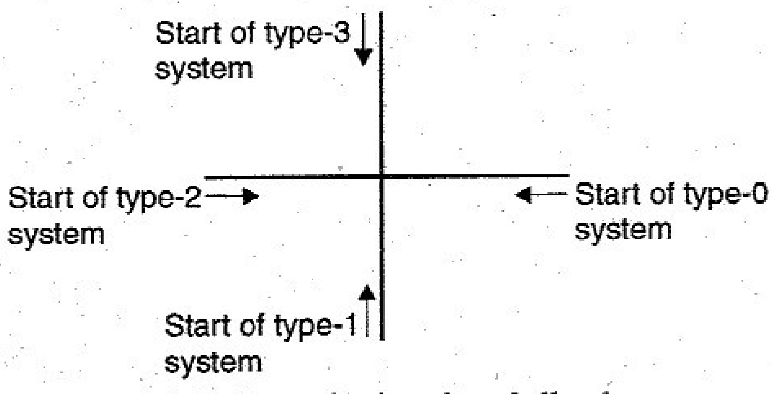


***Fig 3.21 :** Start of polar plot of all pole minimum phase system.*

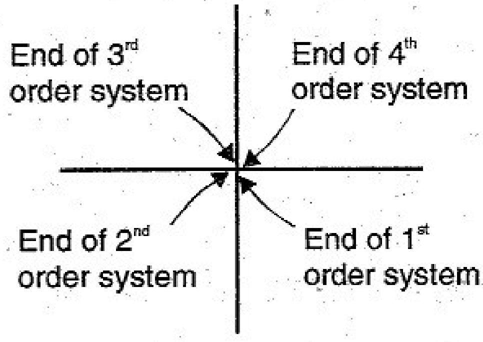


***Fig 3.21 :** Start of polar plot of all pole minimum phase system.*

TABLE-3.1 : Typical Sketches of Polar Plot

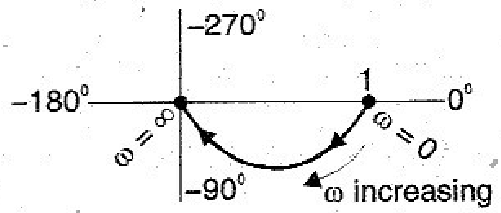
Type : 0, Order : 1

$$G(s) = \frac{1}{1+sT}$$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}\ \angle \tan^{-1}\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1}\omega T$$

As $\omega \to 0$, $\quad G(j\omega) \to 1\angle 0^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -90^\circ$



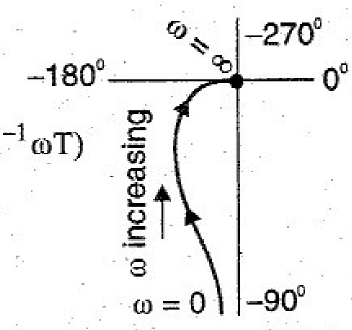
Type : 1, Order : 2

$$G(s) = \frac{1}{s(1+sT)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\ \angle 90^\circ \sqrt{1+\omega^2 T^2}\ \angle \tan^{-1}\omega T} = \frac{1}{\omega\sqrt{1+\omega^2 T^2}} \angle(-90^\circ - \tan^{-1}\omega T)$$

As $\omega \to 0$, $\quad G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -180^\circ$



Type : 0, Order : 2

$$G(s) = \frac{1}{(1+sT_1)(1+sT_2)}$$

$$G(j\omega) = \frac{1}{(1+j\omega T_1)\ (1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2 T_1^2}\ \angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2 T_2^2}\ \angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\sqrt{\left(1+\omega^2 T_1^2\right)\left(1+\omega^2 T_2^2\right)}} \angle(-\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

As $\omega \to 0$, $\quad G(j\omega) \to 1\angle 0^\circ$

As $\omega \to \infty$, $\quad G(j\omega) \to 0\angle -180^\circ$

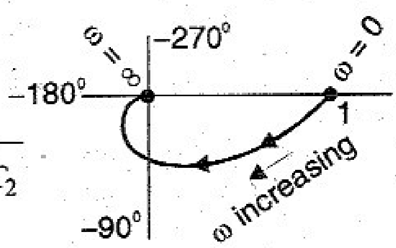
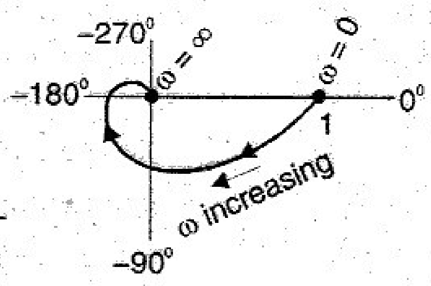


TABLE-3.1 : Typical Sketches of Polar Plot

Type : 0, Order : 3

$$G(s)=\frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$



$$G(j\omega)=\frac{1}{(1+j\omega T_1)\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$=\frac{1}{\sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

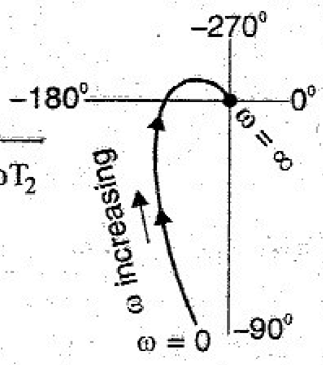
$$=\frac{1}{\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $\quad G(j\omega)\to 1\angle 0^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle-270^\circ$

Type : 1, Order : 3

$$G(s)=\frac{1}{s(1+sT_1)(1+sT_2)}$$



$$G(j\omega)=\frac{1}{j\omega(1+j\omega T_1)\,(1+j\omega T_2)}=\frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

$$=\frac{1}{\omega\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-90^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2)$$

As $\omega\to 0$, $\quad G(j\omega)\to\infty\angle-90^\circ$

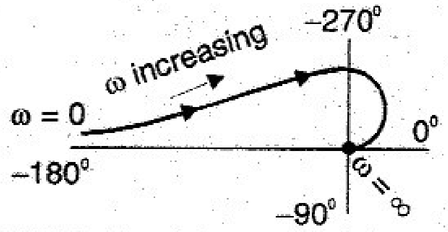
As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle-270^\circ$

Type : 2, Order : 4

$$G(s)=\frac{1}{s^2(1+sT_1)(1+sT_2)}$$

$$G(j\omega)=\frac{1}{(j\omega)^2\ (1+j\omega T_1)\ (1+j\omega T_2)}=\frac{1}{\omega^2\angle-180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2}$$

$$=\frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2)$$

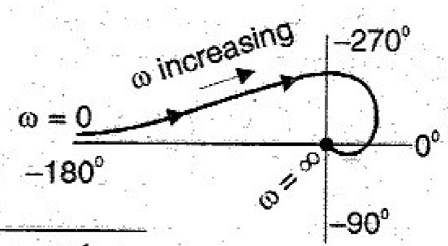


As $\omega\to 0$, $\quad G(j\omega)\to\infty\angle-180^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle-360^\circ$

Type : 2, Order : 5

$$G(s)=\frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$$



$$G(j\omega)=\frac{1}{(j\omega)^2\,(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$=\frac{1}{\omega^2\angle-180^\circ\ \sqrt{1+\omega^2T_1^2}\angle\tan^{-1}\omega T_1\ \sqrt{1+\omega^2T_2^2}\angle\tan^{-1}\omega T_2\ \sqrt{1+\omega^2T_3^2}\angle\tan^{-1}\omega T_3}$$

$$=\frac{1}{\omega^2\sqrt{\left(1+\omega^2T_1^2\right)\left(1+\omega^2T_2^2\right)\left(1+\omega^2T_3^2\right)}}\angle(-180^\circ-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2-\tan^{-1}\omega T_3)$$

As $\omega\to 0$, $\quad G(j\omega)\to\infty\angle-180^\circ$

As $\omega\to\infty$, $\quad G(j\omega)\to 0\angle-450^\circ=0\angle-90^\circ$

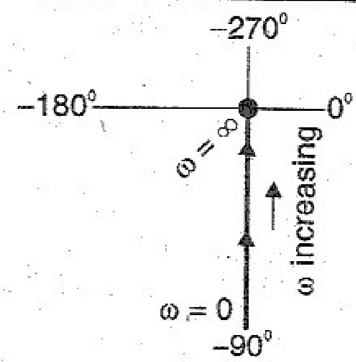
TABLE-3.1 : Typical Sketches of Polar Plot

Type : 1, Order : 1 $G(s)=\frac{1}{s}$

$$G(j\omega)=\frac{1}{j\omega}=\frac{1}{\omega\angle 90^\circ}=\frac{1}{\omega}\angle -90^\circ$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90^\circ$

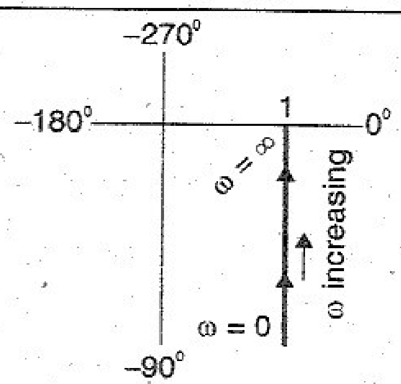


$$G(s)=\frac{1+sT}{sT}$$

$$G(j\omega)=\frac{1+j\omega T}{j\omega T}=\frac{1}{j\omega T}+1=\frac{1}{\omega T\angle 90^\circ}+1=\frac{1}{\omega T}\angle -90^\circ+1$$

As $\omega \to 0$, $G(j\omega) \to \infty\angle -90^\circ+1$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90^\circ+1$

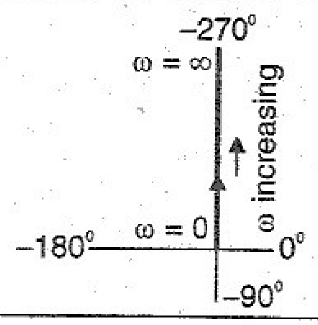


$G(s) = s$

$G(j\omega)=j\omega=\omega\angle 90^\circ$

As $\omega \to 0$, $G(j\omega) \to 0\angle 90^\circ$

As $\omega \to \infty$, $G(j\omega) \to \infty\angle 90^\circ$

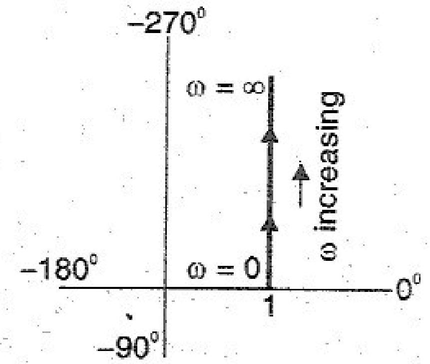


$G(s) = 1+sT$

$G(j\omega)=1+j\omega T=1+\omega T\angle 90^\circ$

As $\omega \to 0$, $G(j\omega) \to 1+0\angle 90^\circ$

As $\omega \to \infty$, $G(j\omega) \to 1+\infty\angle 90^\circ$



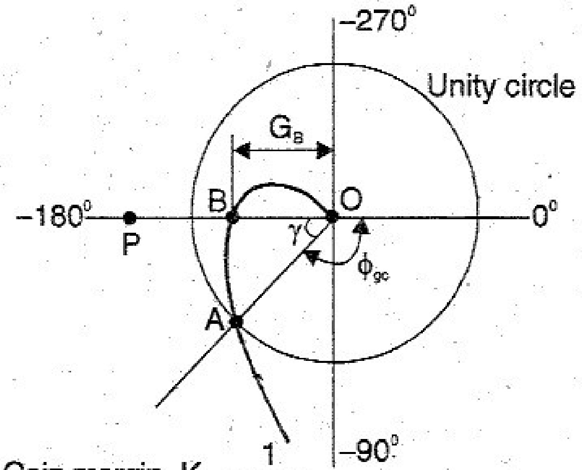
DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM POLAR PLOT

The **gain margin** is defined as the inverse of the magnitude of G(jω) at phase crossover frequency. The **phase crossover frequency** is the frequency at which the phase of G(jω) is 180°.

Let the polar plot cut the 180° axis at point **B** and the magnitude circle passing through the point B be G_B. Now the Gain margin, $K_g = 1/G_B$. If the point B lies within unity circle, then the Gain margin is positive otherwise negative. (If the polar plot is drawn in ordinary graph sheet using rectangular coordinates then the point B is the cutting point of G(jω) locus with negative real axis and $K_g = 1/|G_B|$ where G_B is the magnitude corresponding to point B).

The **phase margin** is defined as, phase margin, $\gamma = 180^0 + \phi_{gc}$ where ϕ_{gc} is the phase angle of G(jω) at gain crossover frequency. The **gain crossover frequency** is the frequency at which the magnitude of G(jω) is unity.

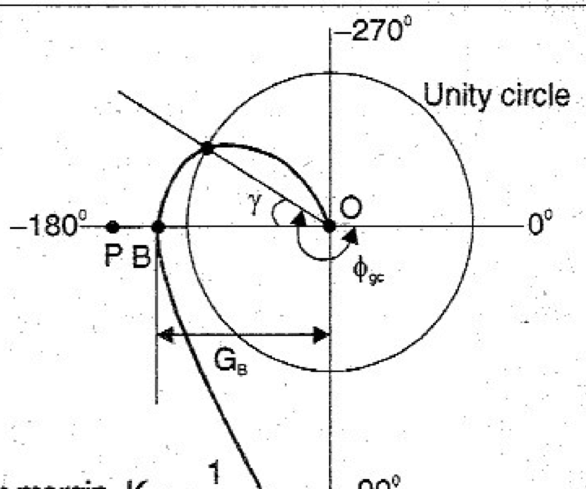
Let the polar plot cut the unity circle at point A as shown in fig 3.23 and 3.24. Now the phase margin, γ is given by $\angle AOP$, i.e. if $\angle AOP$ is below $-180°$ axis then the phase margin is positive and if it is above $-180°$ axis then the phase margin is negative.



Gain margin, $K_g = \frac{1}{G_B}$

Phase margin, $\gamma = 180° + \phi_{gc}$

Fig 3.23 : Polar plot showing positive gain margin and phase margin.



Gain margin, $K_g = \frac{1}{G_B}$

Phase margin, $\gamma = 180° + \phi_{gc}$

Fig 3.24 : Polar plot showing negative gain margin and phase margin.

GAIN ADJUSTMENT USING POLAR PLOT

To Determine K for Specified GM

Draw $G(j\omega)$ locus with K =1. Let it cut the $-180°$ axis at point B corresponding to a gain of G_B. Let the specified gain margin be x db. For this gain margin, the $G(j\omega)$ locus will cut $-180°$ at point A whose magnitude is G_A.

$$\text{Now, } 20\log\frac{1}{G_A} = x \Rightarrow \log\frac{1}{G_A} = \frac{x}{20} \Rightarrow \frac{1}{G_A} = 10^{x/20}$$

$$\therefore G_A = \frac{1}{10^{x/20}}$$

Now the value of K is given by, $K = \frac{G_A}{G_B}$.

If, K >1, then the system gain should be increased.

If, K < 1, then the system gain should be reduced.

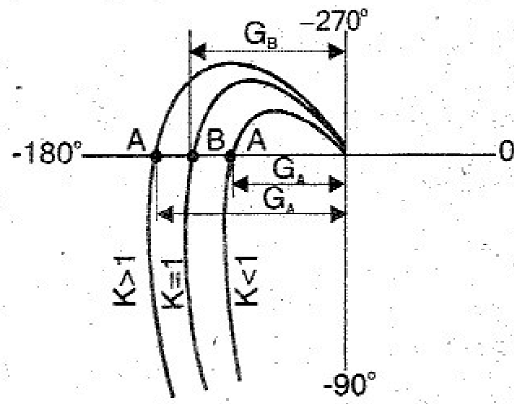


Fig 3.25 : Polar plot for different values of K

To Determine K for Specified PM

Draw $G(j\omega)$ locus with K = 1. Let it cut the unity circle at point B. (The gain at point B is G_B and equal to unity). Let the specified phase margin be x°

For a phase margin of x°, let ϕ_{gcx} be the phase angle of $G(j\omega)$ at gain crossover frequency.

$$\therefore x° = 180° + \phi_{gcx} \Rightarrow \phi_{gcx} = x° - 180°$$

In the polar plot, the radial line corresponding to ϕ_{gcx} will cut the locus of $G(j\omega)$ with K = 1 at point A and the magnitude corresponding to that point be G_A

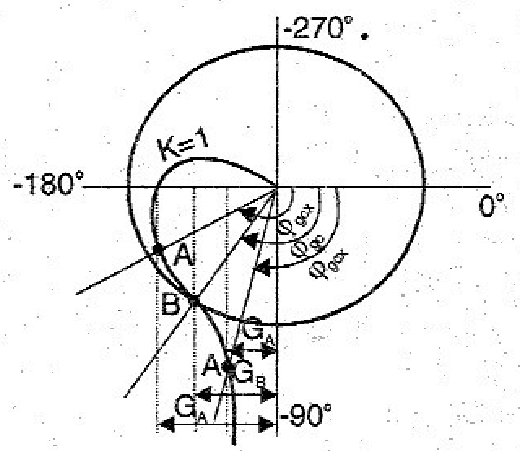


Fig 3.26 : Gain adjustment for required phase margin.

Now, $K = \frac{G_B}{G_A} = \frac{1}{G_A}$ $(\because G_B = 1)$

EXAMPLE 3.7

The open loop transfer function of a unity feedback system is given by G(s) = 1/s(1+s) (1+2s). Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, G(s) = 1/s(1+s) (1+2s)

Put s = jω.

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$$

The corner frequencies are $\omega_{c1} = 1/2 = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and for frequencies around corner frequencies and tabulated in table-1. Using polar to rectangular conversion, the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.7.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 3.7.2.

$$G(j\omega) = \frac{1}{(j\omega)\,(1+j\omega)\,(1+j2\omega)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)\,(1+4\omega^2)}}\angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)\,(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

TABLE-1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
|G(jω)|	2.2	1.8	1.5	1.2	0.9	0.7	0.3
∠G(jω) deg	-144	-150	-156	-162	-171	-179.5 ≈ -180	-198

TABLE-2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

RESULT

Gain margin, K_g = 1.4286

Phase margin, γ = +12°

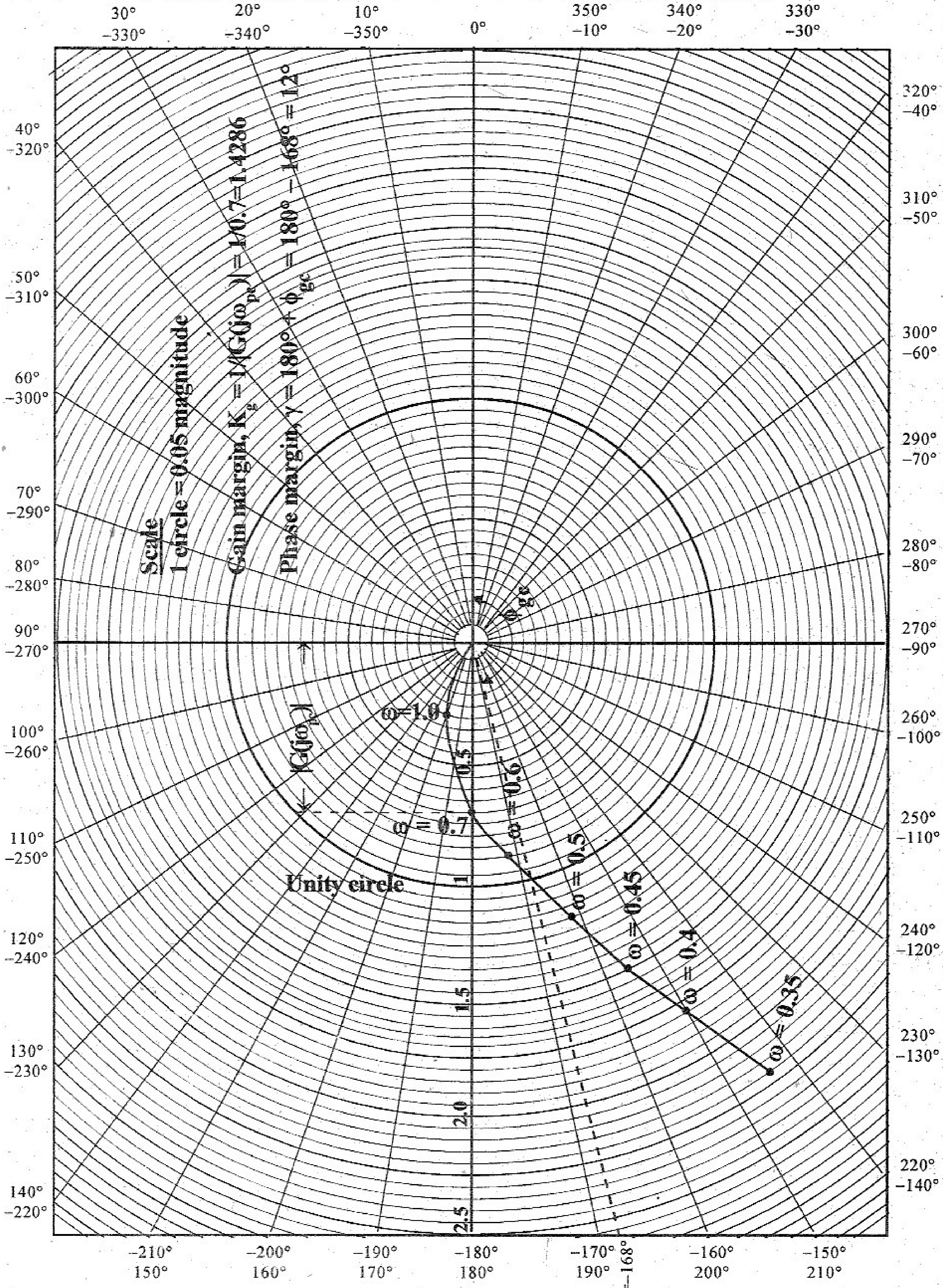


Fig 3.7.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)\,(1+j2\omega)$ (using polar coordinates).

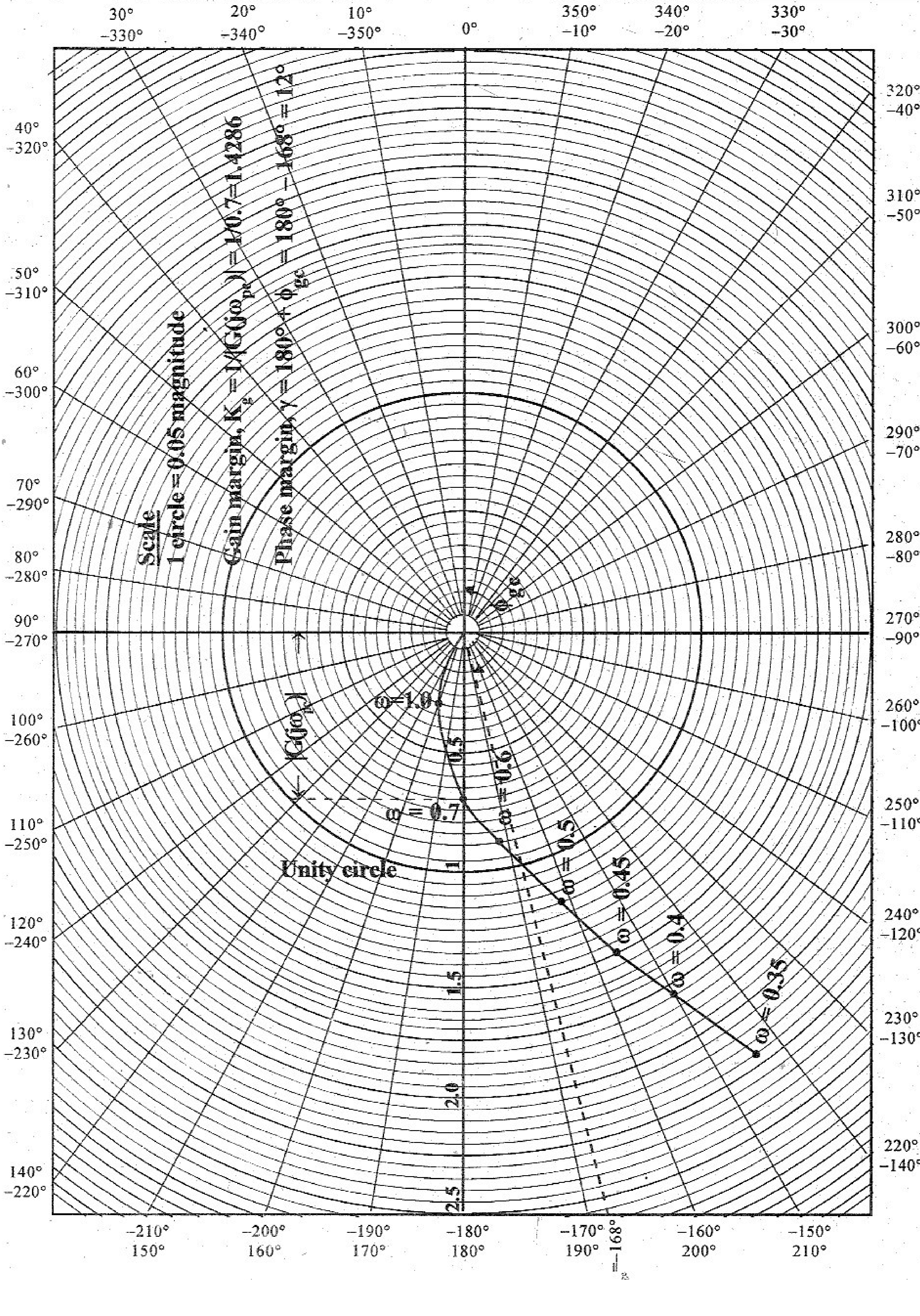


Fig 3.7.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j\omega)\,(1+j2\omega)$ (using polar coordinates).

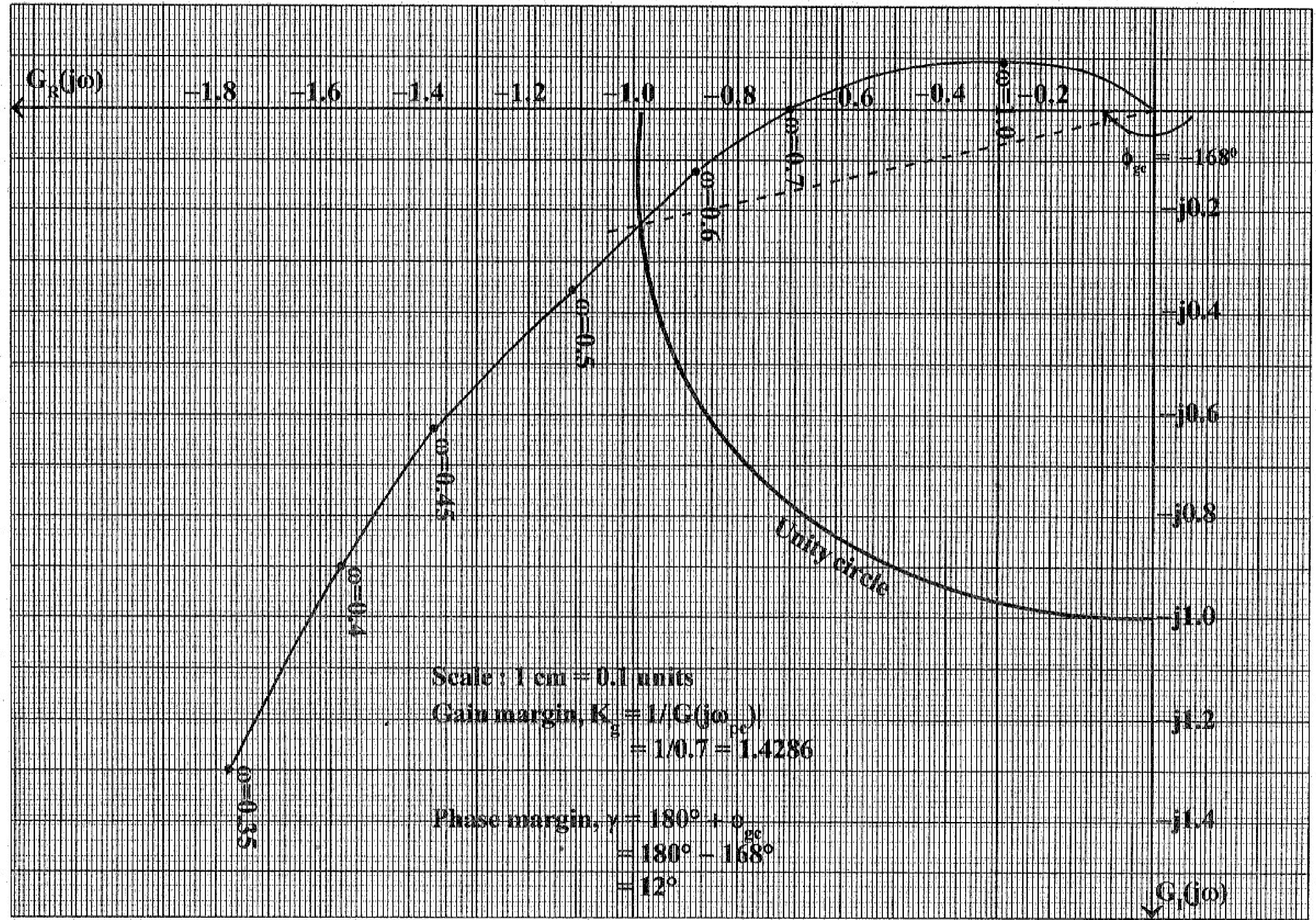


Fig 3.7.2: Polar plot of, $G(j\omega) = \dfrac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$ *(using rectangular coordinates).*

EXAMPLE 3.8

The open loop transfer function of a unity feedback system is given by $G(s) = 1/\ s^2(1+s)\ (1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, $G(s) = 1/\ s^2(1+s)\ (1+2s)$

Put $s = j\omega$, $\therefore\ G(j\omega) = \dfrac{1}{(j\omega)^2\ (1+j\omega)\ (1+j2\omega)}$

The corner frequencies are $\omega_{c1} = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec. The magnitude and phase angle of $G(j\omega)$ are calculated for the corner frequencies and frequencies around corner frequencies are tabulated in table-1. Using the polar to rectangular conversion, the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.8.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 3.8.2.

$$G(j\omega) = \frac{1}{(j\omega)^2\ (1+j\omega)\ (1+j2\omega)}$$

$$= \frac{1}{\omega^2\angle 180^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\ \angle(-180-\tan^{-1}\omega-\tan^{-1}2\omega)$$

$$|G(j\omega)| = \frac{1}{\omega^2\ \sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)\ (1+4\omega^2)}}$$

$$= \frac{1}{\omega^2\ \sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega.$$

TABLE-1 : Magnitude and phase plot of $G(j\omega)$ at various frequencies

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$\|G(j\omega)\|$	3.3	2.5	1.9	1.5	1.2	$0.97 \approx 1$	0.8	0.3
$\angle G(j\omega)$ deg	–246	–251	–256	–261	–265	–269	–273	–288

TABLE-2 : Real and imaginary parts of $G(j\omega)$

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$G_R(j\omega)$	–1.34	–0.81	–0.46	–0.23	–0.1	–0.02	0.04	0.09
$G_I(j\omega)$	3.01	2.36	1.84	1.48	1.2	1.0	0.8	0.29

RESULT

Gain margin, $K_g = 0$

Phase margin, $\gamma = -90^\circ$

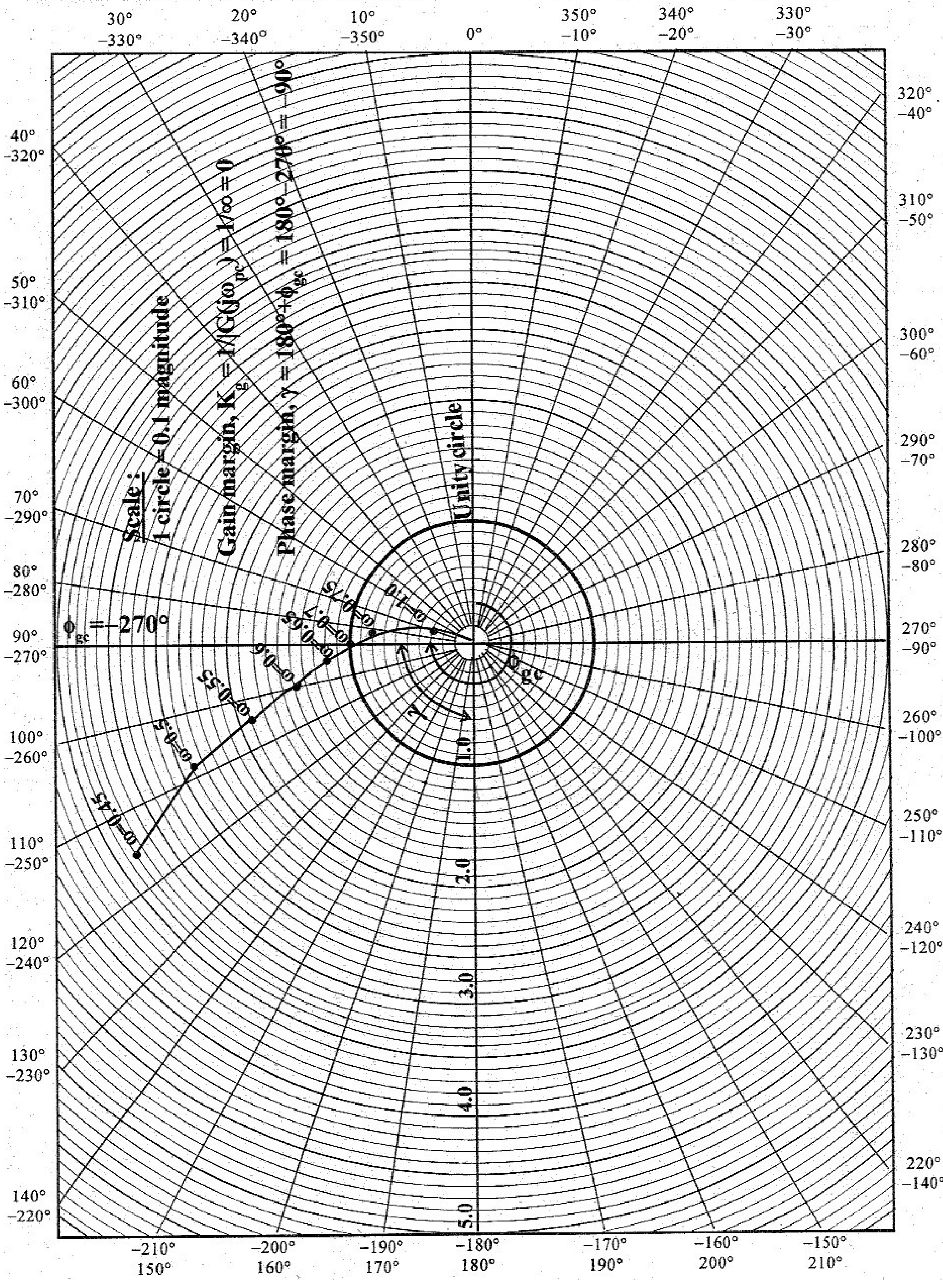


Fig 3.8.1: Polar plot of $G(j\omega) = 1/(j\omega)^2 (1+j\omega)(1+j2\omega)$, *(using polar coordinates)*

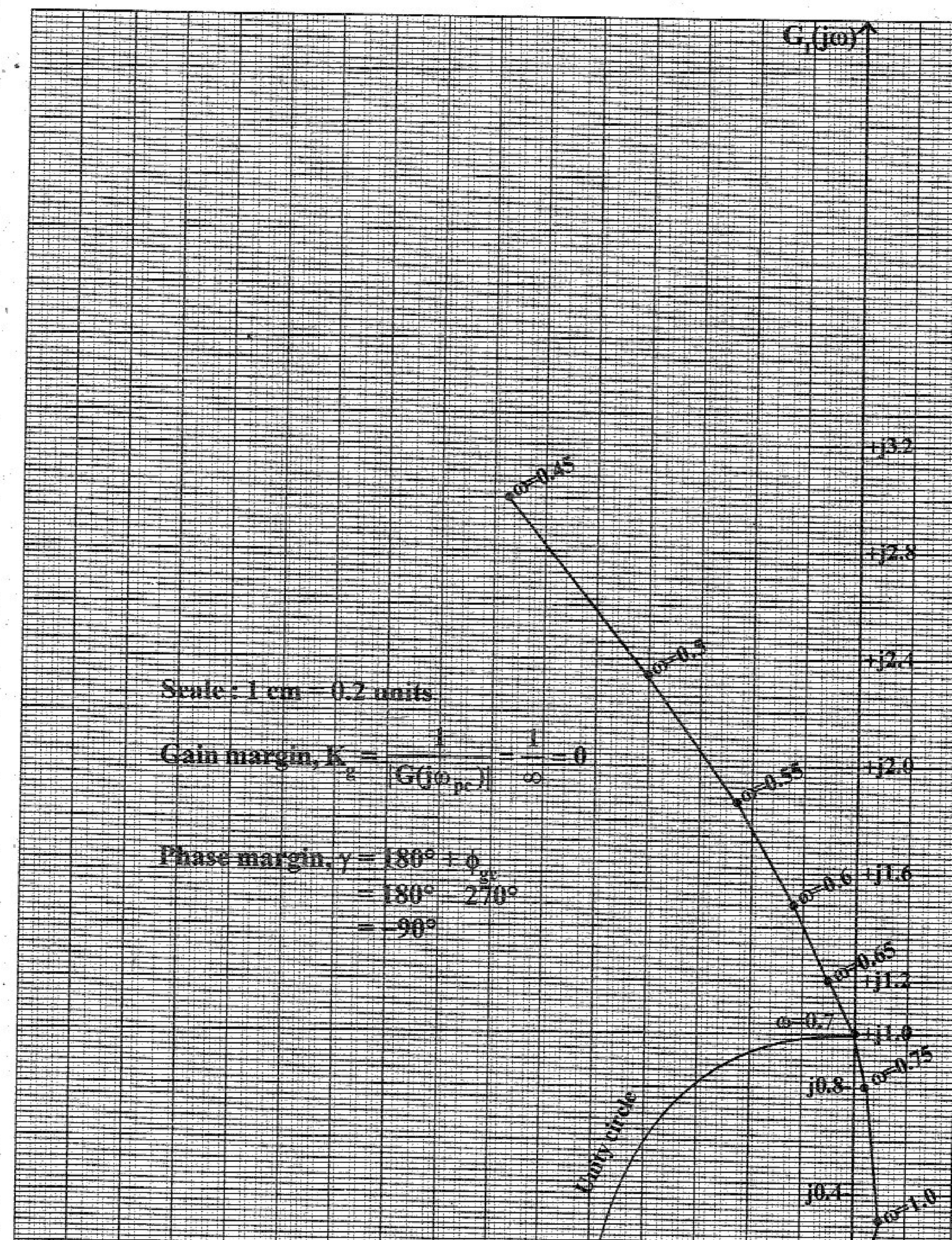


Fig 3.8.2: *Polar plot of* $G(j\omega) = 1/(j\omega)^2\,(1+j\omega)\,(1+j2\omega)$, *(using rectangular coordinates)*

EXAMPLE 3.9

The open loop transfer function of a unity feedback system is given by,

$$G(s) = \frac{(1+0.2s)\,(1+0.025s)}{s^3(1+0.005s)\,(1+0.001s)}\ .$$

Sketch the polar plot and determine the phase margin.

SOLUTION

$$\text{Given that } G(s) = \frac{(1+0.2s)\,(1+0.025s)}{s^3(1+0.005s)\,(1+0.001s)}$$

$$\therefore G(j\omega) = \frac{(1+j0.2\omega)\,(1+j0.025\omega)}{(j\omega)^3\,(1+j0.005\omega)\,(1+j0.001\omega)}$$

$$= \frac{\sqrt{1+(0.2\omega)^2}\ \angle\tan^{-1}0.2\omega\ \sqrt{1+(0.025\omega)^2}\ \angle\tan^{-1}0.025\omega}{\omega^3\ \angle 270^\circ\ \sqrt{1+(0.005\omega)^2}\ \angle\tan^{-1}0.005\omega\ \sqrt{1+(0.001\omega)^2}\ \angle\tan^{-1}0.001\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.025\omega)^2}}{\omega^3\ \sqrt{1+(0.005\omega)^2}\ \sqrt{1+(0.001\omega)^2}}$$

$$\angle G(j\omega) = \tan^{-1}0.2\omega + \tan^{-1}0.025\omega - 270^\circ - \tan^{-1}0.005\omega - \tan^{-1}0.001\omega$$

The magnitude and phase angle of $G(j\omega)$ are calculated for various frequencies and listed in table-1. Using the polar to rectangular conversion, the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.9.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 3.9.2.

TABLE-1 : Magnitude and phase of $G(j\omega)$

ω,rad/sec	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$\lvert G(j\omega)\rvert$	1.4	1.2	1.0	0.8	0.6	0.4	0.2
$\angle G(j\omega)$, deg	−259	−258	−257	−256	−255	−253	−249

TABLE-2 : Real and imaginary part of $G(j\omega)$

ω, rad/sec	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$G_R(j\omega)$	−0.27	−0.25	−0.22	−0.19	−0.16	−0.12	−0.07
$G_I(j\omega)$	1.37	1.17	0.97	0.78	0.58	0.38	0.19

RESULT

Phase margin, $\gamma = -77^\circ$

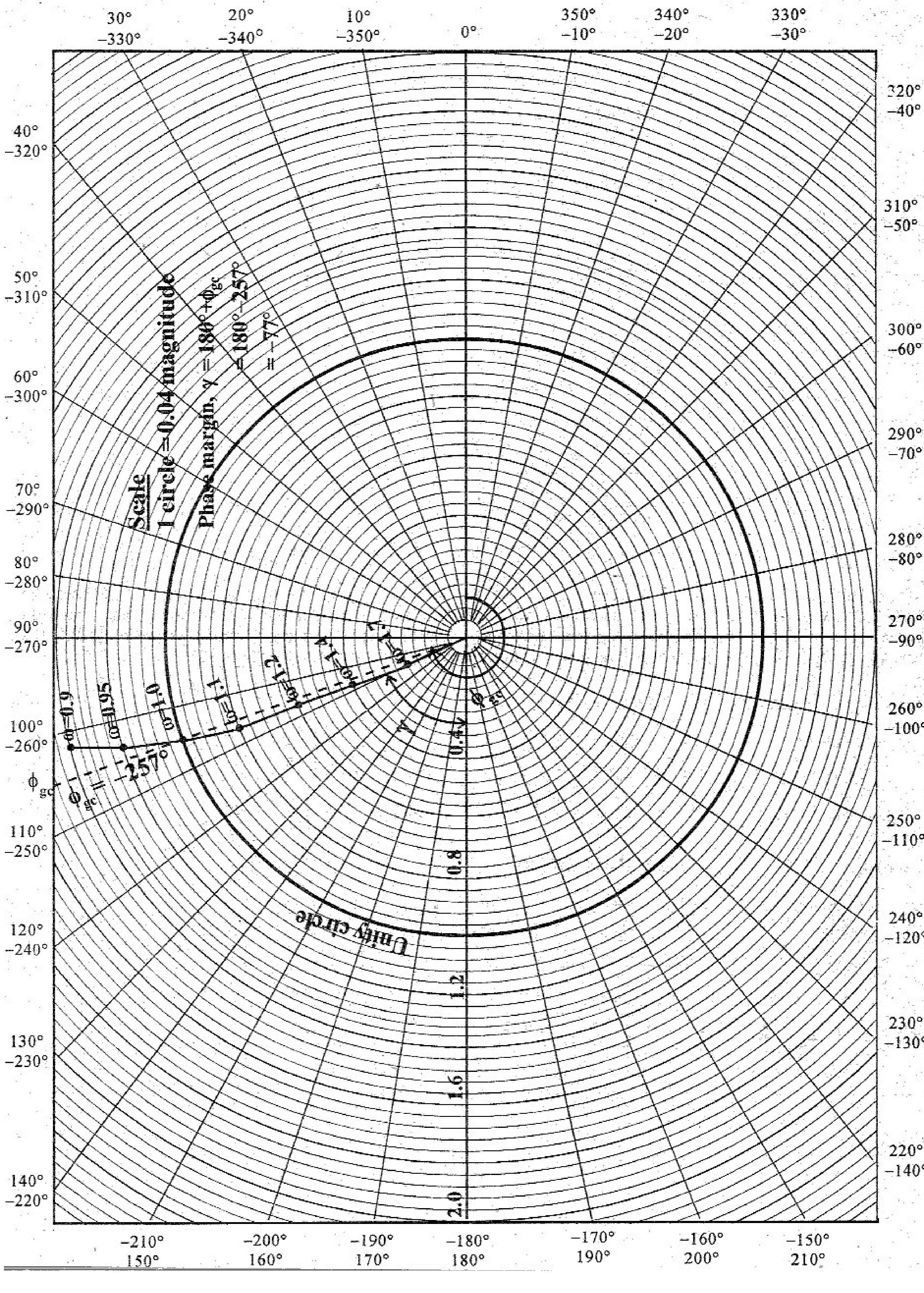


Fig 3.9.1: Polar plot of $G(j\omega) = (1+j0.2\omega)(1+j0.025\omega) / (j\omega)^3(1+j0.005\omega)(1+j0.001\omega)$, *(using polar coordinates)*

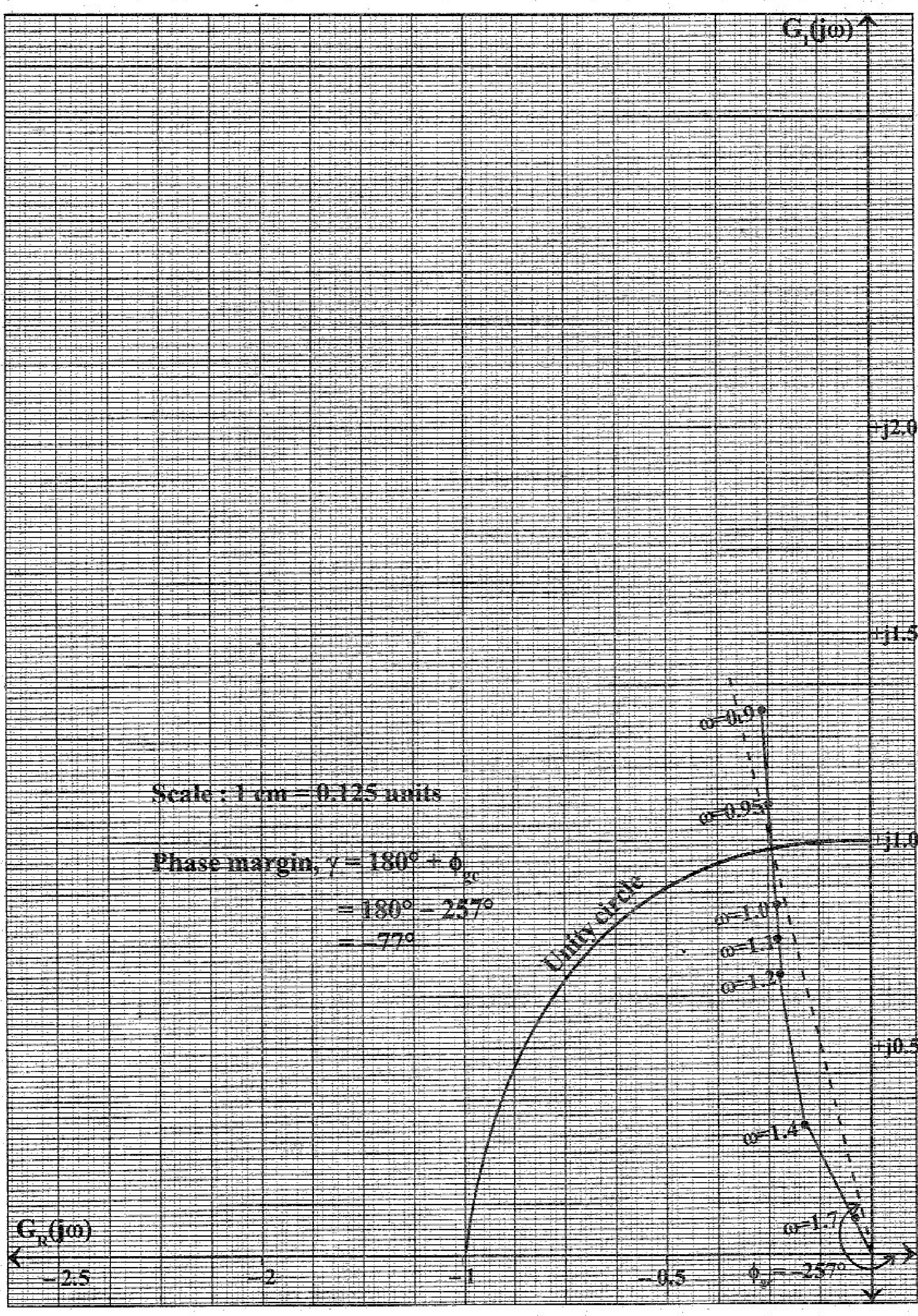


Fig 3.9.2: Polar plot of $G(j\omega) = (1+j0.2\omega)(1+j0.025\omega) / (j\omega)^3(1+j0.005\omega)(1+j0.001\omega)$, *(using rectangular coordinates)*

EXAMPLE 3.10

The open loop transfer function of a unity feedback system is given by G(s) = 1/s(1+s)². Sketch the polar plot and determine the gain and phase margin.

SOLUTION

Given that, G(s) = 1/s(1+s)².

Put s = jω,

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j\omega)^2} = \frac{1}{j\omega\,(1+j\omega)\,(1+j\omega)}$$

The corner frequency is ω_{c1} = 1 rad/sec. The magnitude and phase angle of G(jω) are calculated for corner frequency and frequencies around corner frequency and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.10.1. The polar plot using rectangular coordinates are sketched on an ordinary graph sheet as shown in fig 3.10.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega)^2} = \frac{1}{j\omega\,(1+j\omega)\,(1+j\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega}$$

$$= \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2}\quad \angle(-90^\circ - 2\tan^{-1}\omega)$$

$$|G(j\omega)| = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}\omega$$

TABLE-1: Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
|G(jω)|	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
∠G(jω) deg	−134	−143	−151	−159	−167	−174	−180	−185

TABLE-2 : Real and imaginary parts of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$G_R(j\omega)$	−1.53	−1.28	−1.05	−0.93	−0.78	−0.6	−0.5	−0.4
$G_I(j\omega)$	−1.58	−0.96	−0.58	−0.36	−0.18	0.06	0	0.03

RESULT

Gain margin, $K_g = 2$

Phase margin, $\gamma = 21°$

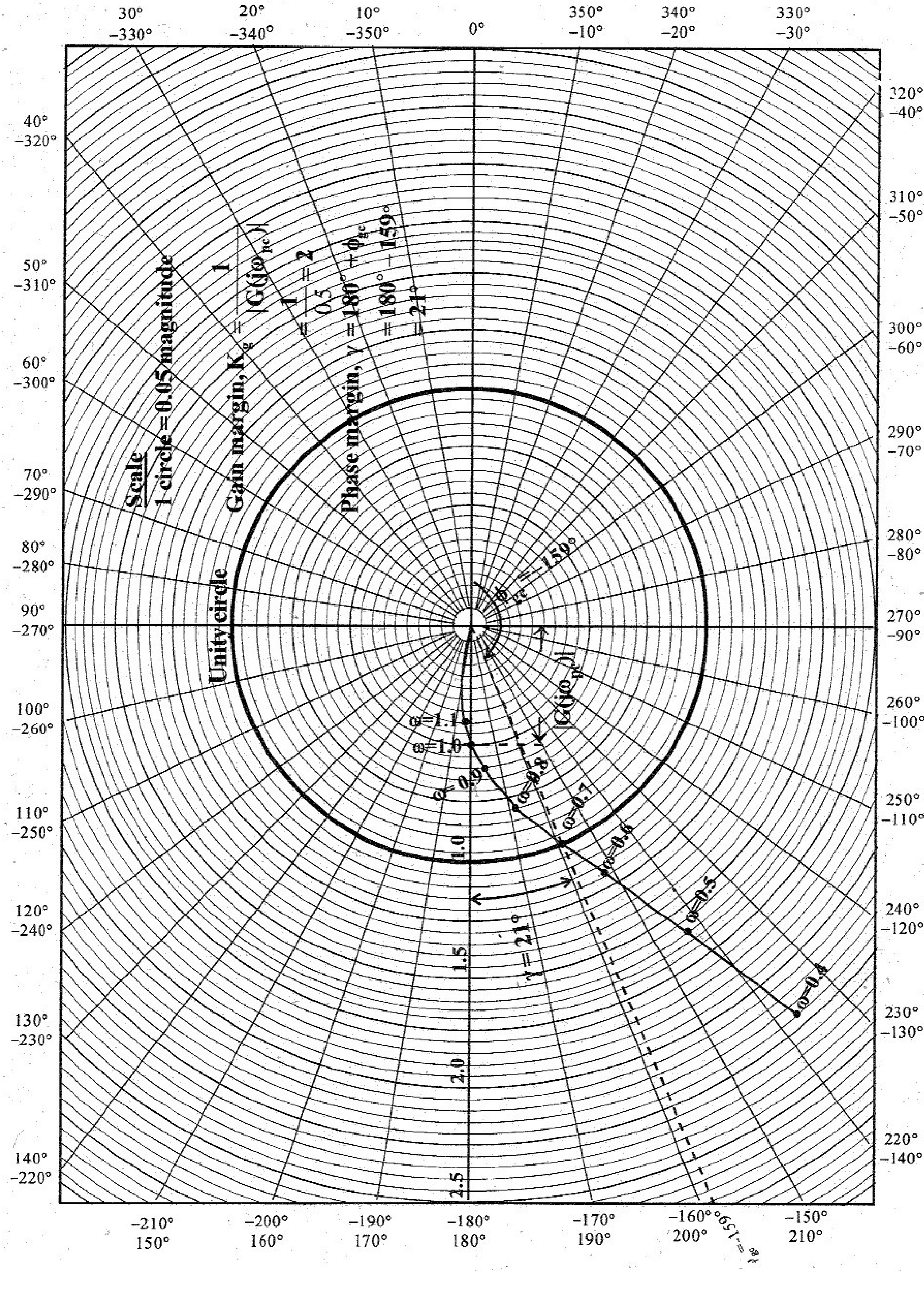


Fig 3.10.1: Polar plot of $G(j\omega) = 1/[j\omega\,(1+j\omega)^2]$ (using polar coordinates)

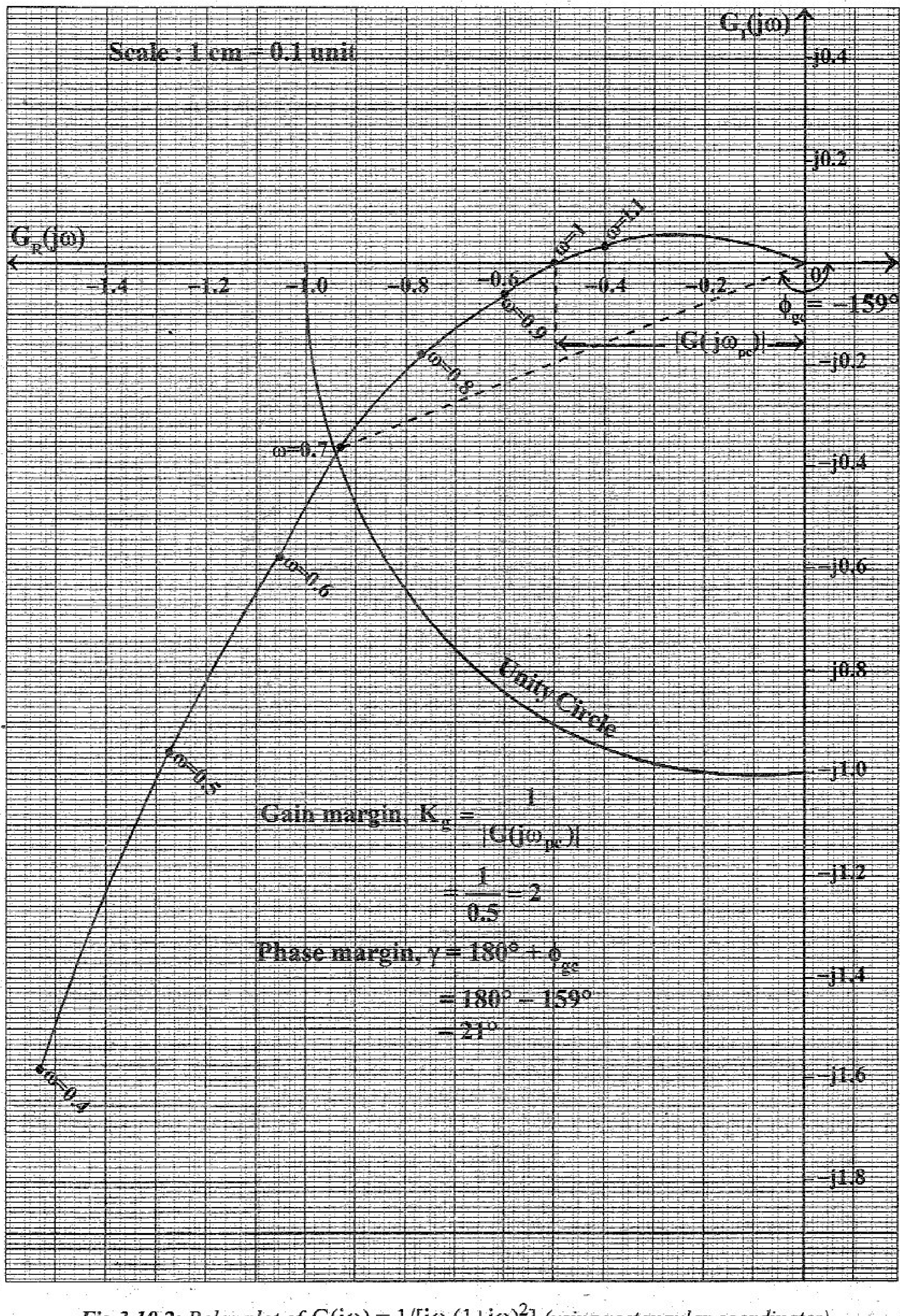


Fig 3.10.2: Polar plot of $G(j\omega) = 1/[j\omega\,(1+j\omega)^2]$ *(using rectangular coordinates)*

EXAMPLE 3.11

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)\,(1+0.05s)}$.

Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db (ii) Phase margin is 60°.

SOLUTION

Given that, $G(s) = \frac{K}{s\,(1+0.2s)\,(1+0.05s)}$. The polar plot is sketched by taking K = 1.

$\therefore$ Put K = 1 and s = jω in G(s). $\therefore\ G(j\omega) = \frac{1}{j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)}$

The corner frequencies are $\omega_{c1} = 1/0.2 = 5$ rad/sec and $\omega_{c2} = 1/0.05 = 20$ rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.11.1. Polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 3.11.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+(0.2\omega)^2}\ \angle\tan^{-1}0.2\omega\ \sqrt{1+(0.05\omega)^2}\ \angle\tan^{-1}0.05\omega}$$

$$= \frac{1}{\omega\ \sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}}\ \angle(-90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}} \quad \text{and} \quad \angle G(j\omega) = -90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

TABLE-1 : Magnitude and Phase of G(jω) at Various Frequencies

ω rad/sec	0.6	0.8	1	2	3	4
|G(jω)|	1.65	1.23	1.0	0.5	0.3	0.2
∠G(jω) deg	−98	−101	−104	−117.5	−129.4	−140

ω rad/sec	5	6	7	9	10	11	14
|G(jω)|	0.14	0.1	0.07	0.05	0.04	0.03	0.02
∠G(jω) deg	−149	−157	−164	−176	−180	−184	−195

TABLE-2 : Real and Imaginary Parts of G(jω) at Various Frequencies

ω rad/sec	0.6	0.8	1	2	3	4
$G_R(j\omega)$	−0.23	−0.23	−0.24	−0.23	−0.19	−0.15
$G_I(j\omega)$	−1.63	−1.21	−0.97	−0.44	−0.23	−0.13

ω rad/sec	5	6	7	9	10	11	14
$G_R(j\omega)$	–0.120	–0.092	–0.067	–0.050	–0.04	–0.030	–0.019
$G_I(j\omega)$	–0.072	–0.039	–0.019	–0.0034	0	0.002	0.005

In the polar plot shown in fig 3.11.1 and 3.11.2 there are two plots, marked as curve-I and curve-II. These two loci are sketched with different scales to clearly determine the gain margin and phase margin.

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.04 = 25$.

Gain margin in db = 20 log 25 = 28 db.

Phase margin, $\gamma = 76°$.

Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. From the polar plot $G_B = 0.04$. The gain margin of 28 db with K = 1 has to be reduced to 18 db and so K has to be increased to a value greater than one.

Let G_A be the gain at –180° for a gain margin of 18 db.

$$\text{Now, } 20\log\frac{1}{G_A} = 18 \quad\Rightarrow\quad \log\frac{1}{G_A} = \frac{18}{20} \quad\Rightarrow\quad \frac{1}{G_A} = 10^{18/20}$$

$$\therefore G_A = \frac{1}{10^{18/20}} = 0.125$$

$$\text{The value of K is given by, } K = \frac{G_A}{G_B} = \frac{0.125}{0.04} = 3.125$$

Case (ii)

With K = 1, the phase margin is 76°. This has to be reduced to 60°. Hence gain has to be increased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 60°

$$\therefore 60° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 60° - 180° = -120°$$

In the polar plot the –120° line cut the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 0.425$ and $G_D = 1$.

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{0.425} = 2.353$$

RESULT

(a) When K = 1, Gain margin, K_g = 25

Gain margin in db = 28db

(b) When K = 1, Phase margin, γ = 76°

(c) For a gain margin of 18 db, K = 3.125

(d) For a phase margin of 60°, K = 2.353

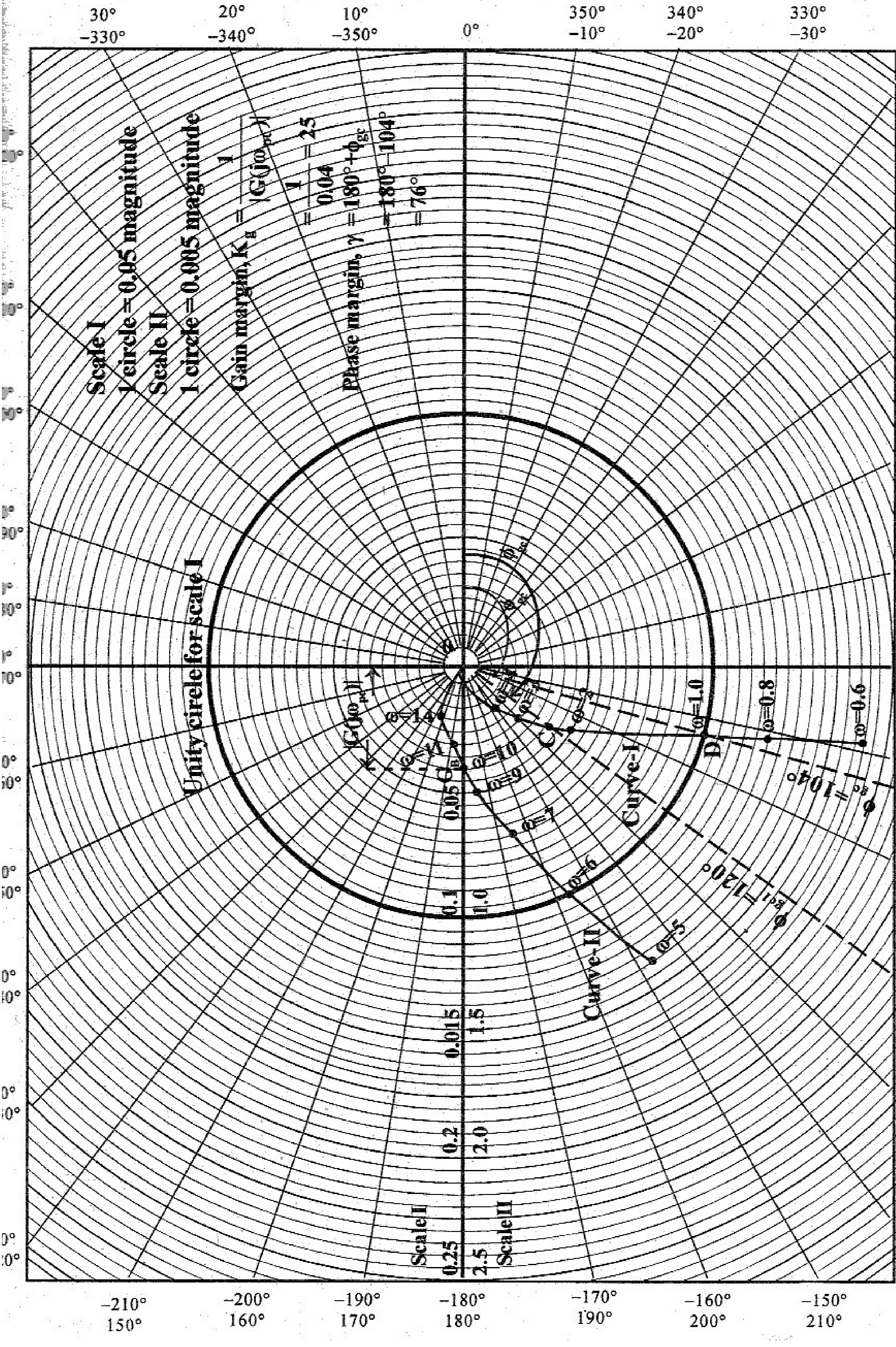


Fig 3.11.1: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)$, *(using polar coordinates)*

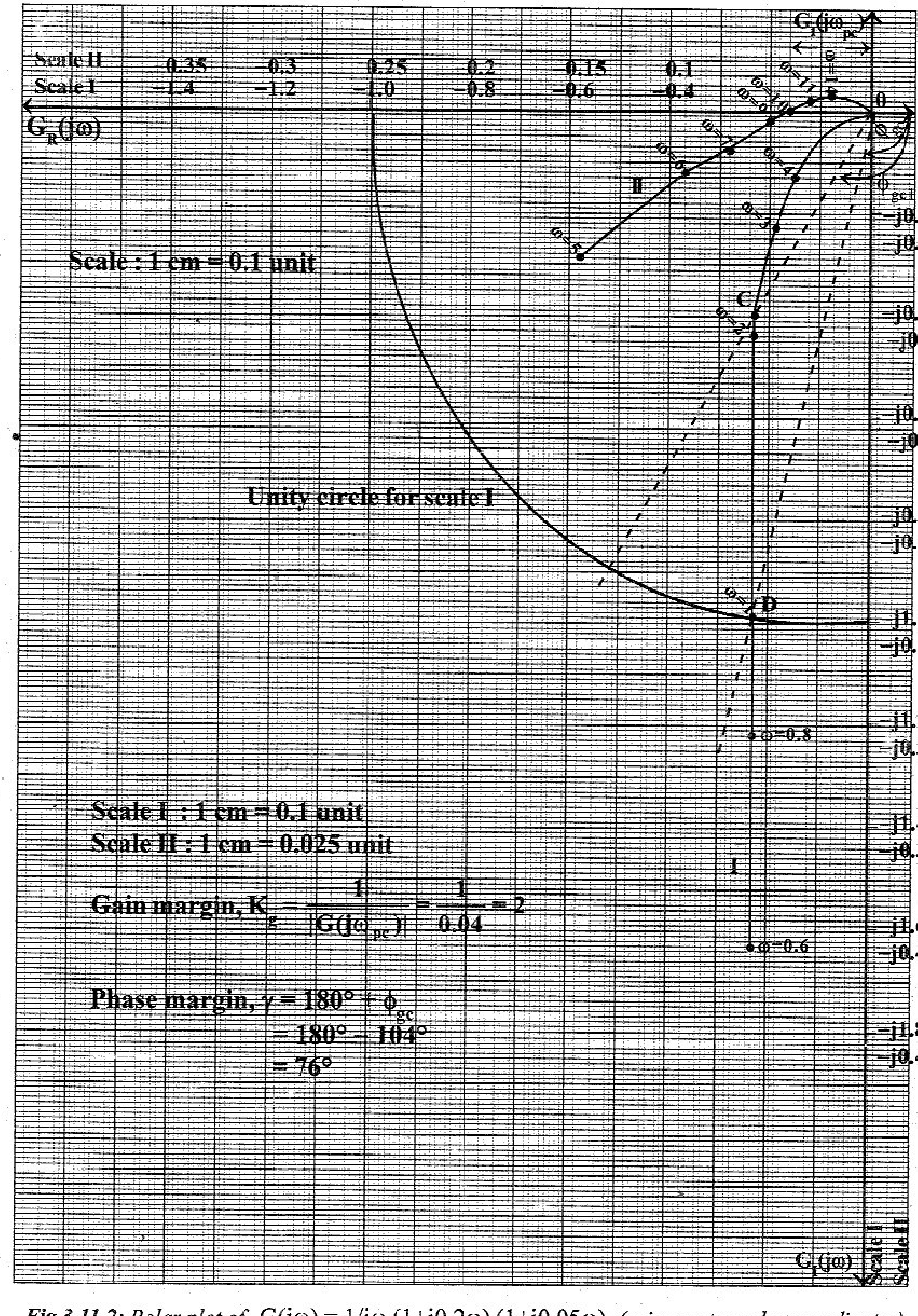


Fig 3.11.2: Polar plot of $G(j\omega) = 1/j\omega\,(1+j0.2\omega)\,(1+j0.05\omega)$, *(using rectangular coordinates)*

EXAMPLE 3.12

Consider a unity feedback system having an open loop transfer function, $G(s) = \dfrac{K}{s\,(1+0.5s)\,(1+4s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 20 db and (ii) Phase margin is 30°.

SOLUTION

Given that, G(s) = K/s (1+0.5s) (1+4s)

The polar plot is sketched by taking K = 1.

Put K = 1 and s = jω in G(s).

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

The corner frequencies are ω_{c1} = 1/4 = 0.25 rad/sec and ω_{c2} = 1/0.5 = 2 rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 3.12.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 3.12.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+(0.5\omega)^2}\ \angle\tan^{-1}0.5\omega\ \sqrt{1+(4\omega)^2}\ \angle\tan^{-1}4\omega}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}\ \angle(-90^\circ-\tan^{-1}0.5\omega-\tan^{-1}4\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega$$

TABLE-1 : Magnitude and Phase of G(jω) at Various Frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
|G(jω)|	2.11	1.3	0.87	0.61	0.35	0.22	0.15
∠G(jω) deg	−149	−159	−167	−174	−184	−193	−199

TABLE-2 : Real part and Imaginary parts of G(jω) at Various Frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$G_R(j\omega)$	−1.8	−1.21	−0.85	−0.61	−0.35	−0.21	−0.14
$G_I(j\omega)$	−1.09	−0.47	−0.2	−0.06	0.02	0.05	0.05

From the polar plot, with K = 1,

Gain margin, K_g = 1/0.44 = 2.27

Gain margin in db = 20 log 2.27 = 7.12 db

Phase margin, $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 165^\circ = 15^\circ$

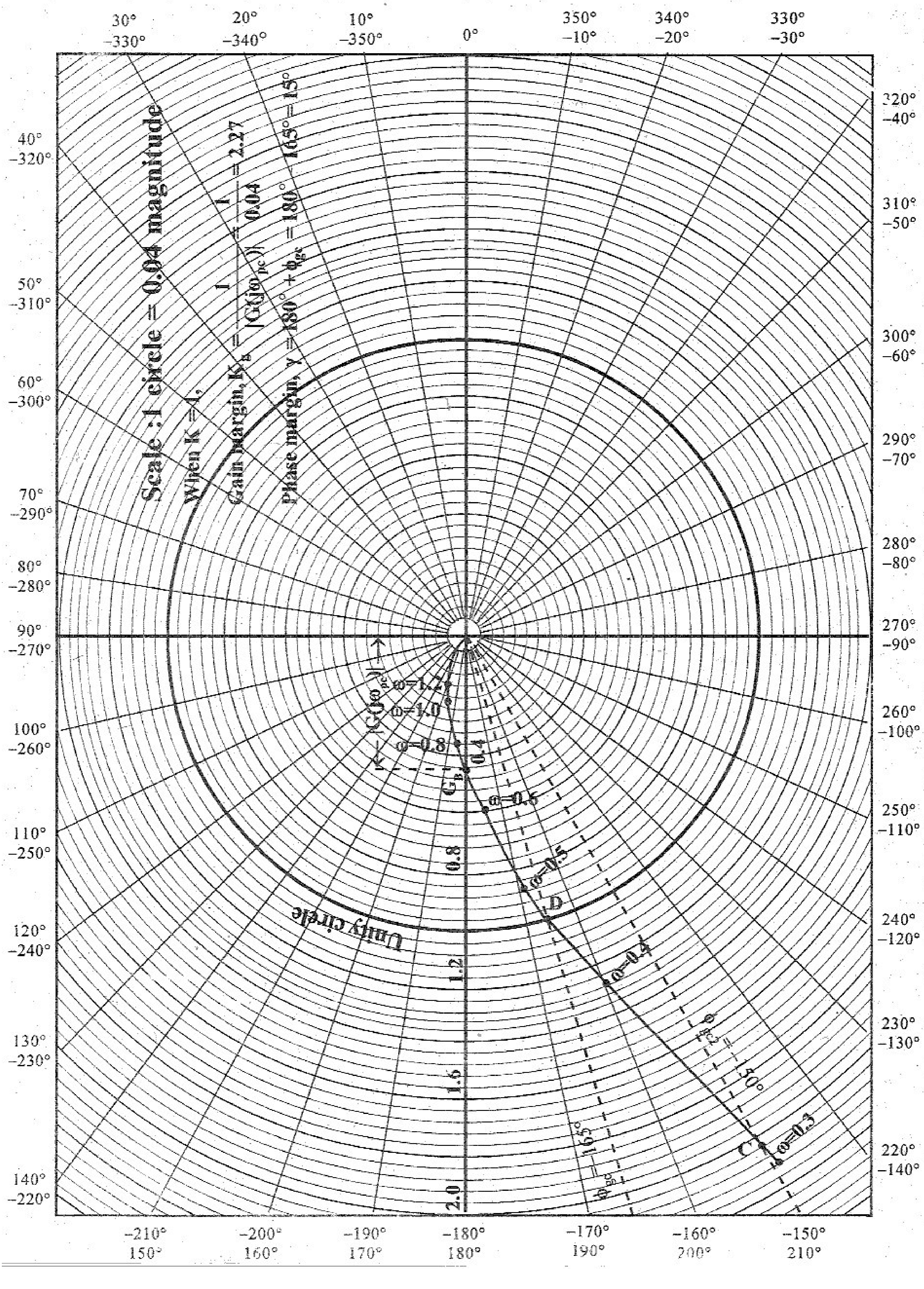


Fig 3.12.1: Polar plot of $G(j\omega) = 1/[j\omega\,(1+j0.5\omega)\,(1+j4\omega)]$, (using polar coordinates)

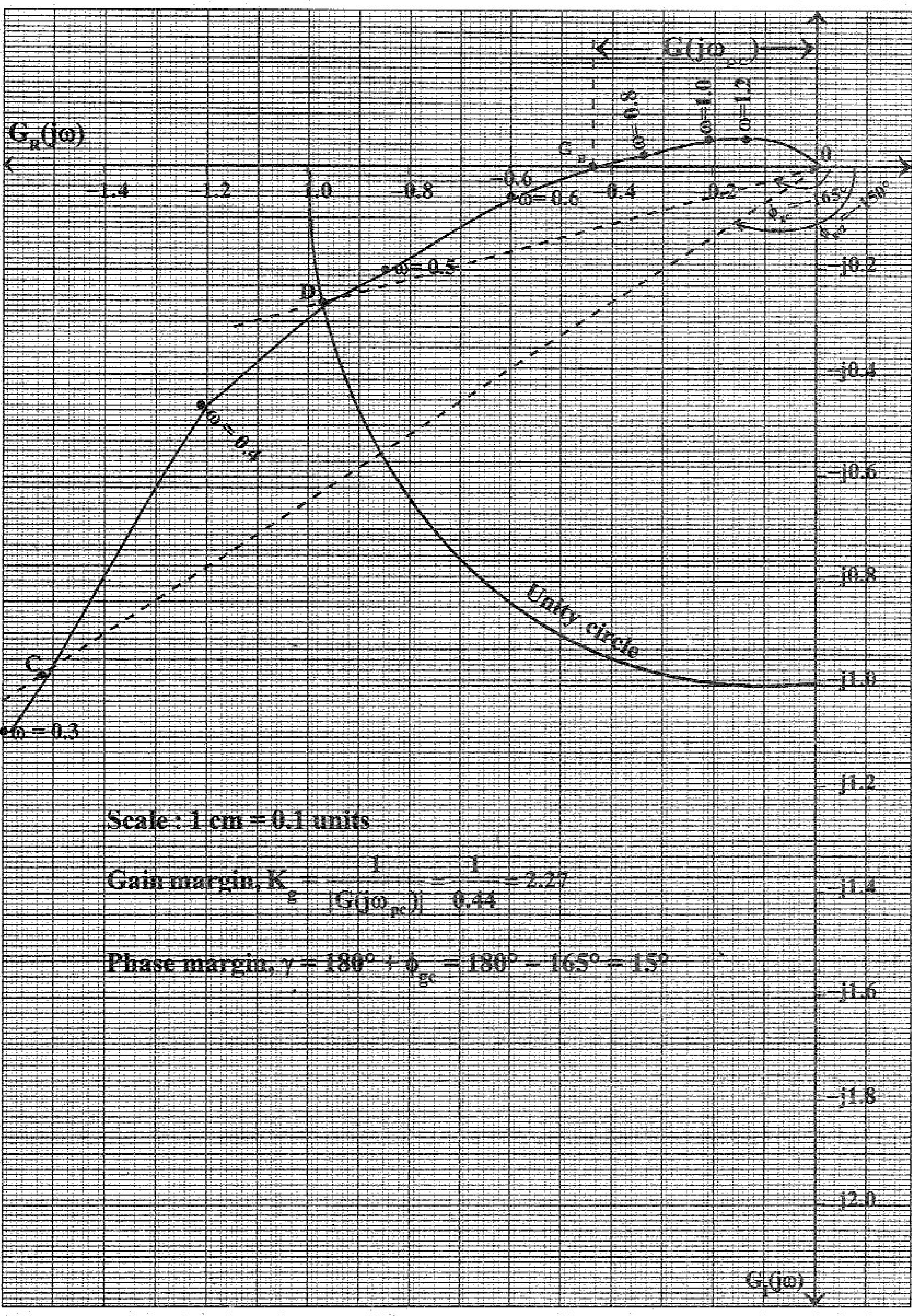


Fig 3.12.2: *Polar plot of* $G(j\omega) = 1/[j\omega\,(1+j0.5\omega)\,(1+j4\omega)]$, *(using rectangular coordinates).*

Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. From the polar plot, $G_B = 0.44$. The gain margin of 7.12 db with K = 1 has to be increased to 20 db and so K has to be decreased to a value less than one.

Let G_A be the gain at –180° for a gain margin of 20 db.

$$\text{Now, } 20 \log \frac{1}{G_A} = 20$$

$$\log \frac{1}{G_A} = \frac{20}{20} = 1$$

$$\frac{1}{G_A} = 10^1 = 10$$

$$\therefore G_A = \frac{1}{10} = 0.1$$

The value of K is given by, $K = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

Case (ii)

With K = 1, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°.

$$\therefore 30° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 30° - 180° = -150°$$

In the polar plot the –150° line cuts the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 2.04$ and $G_D = 1$

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{2.04} = 0.49$$

RESULT

(a) When K = 1, Gain margin, K_g = 2.27

Gain margin in db = 7.12 db

(b) When K = 1, Phase margin, γ = 15°

(c) For a gain margin of 20 db, K = 0.227

(d) For a phase margin of 30°, K = 0.49

3.8 NICHOLS PLOT

The **Nichols plot** is a frequency response plot of the open loop transfer function of a system. The Nichols plot is a graph between magnitude of G(jω) in db and the phase of G(jω) in degree, plotted on a ordinary graph sheet.

In order to plot the Nichols plot, the magnitude of G(jω) in db and phase of G(jω) in deg are computed for various values of ω and tabulated. Usually the choice of frequencies are corner frequencies. Choose appropriate scales for magnitude on y–axis and phase on x–axis. Fix all the points on ordinary graph sheet and join the points by smooth curve, and mark frequencies corresponding to each point.

In another method, first the Bode plot of $G(j\omega)$ is sketched. From the Bode plot the magnitude and phase for various values of frequency, ω are noted and tabulated. Using these values the Nichols plot is sketched as explained earlier.

DETERMINATION OF GAIN MARGIN AND PHASE MARGIN FROM NICHOLS PLOT

The gain margin in db is given by the negative of db magnitude of $G(j\omega)$ at the phase crossover frequency, ω_{pc}. The ω_{pc} is the frequency at which phase of $G(j\omega)$ is $-180°$. If the db magnitude of $G(j\omega)$ at ω_{pc} is negative then gain margin is positive and vice versa.

Let ϕ_{gc} be the phase angle of $G(j\omega)$ at gain cross over frequency ω_{gc}. The ω_{gc} is the frequency at which the db magnitude of $G(j\omega)$ is zero. Now the phase margin, γ is given by $\gamma = 180° + \phi_{gc}$. If ϕ_{gc} is less negative than $-180°$ then phase margin is positive and vice versa. The positive and negative gain margins are illustrated in fig 3.27.

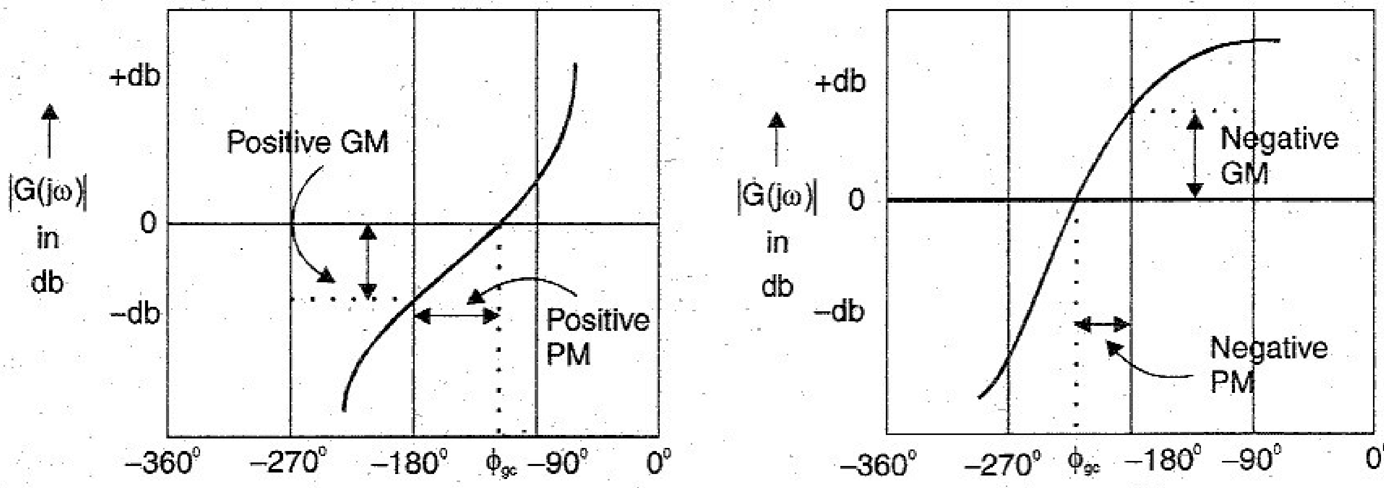


Fig 3.27 : Nichols plot showing phase margin (PM) and gain margin (GM).

GAIN ADJUSTMENT IN NICHOLS PLOT

In the open loop transfer function, $G(j\omega)$ the constant K contributes only magnitude. Hence by changing the value of K the system gain can be adjusted to meet the desired specifications. The desired specifications are gain margin and phase margin.

In a system transfer function, if the value of K required to be estimated, in order to satisfy a desired specification, then draw the Nichols plot of the system with K=1. The constant K can add 20logK to every point of the plot. Due to this addition, the Nichols plot will shift vertically up or down. Hence shift the plot vertically up or down to meet the desired specification. Equate the vertical distance by which the Nichols plot is shifted to 20logK and solve for K.

Let, x = change in db (x is positive if the plot is shifted up and vice versa).

Now, $20 \log K = x \quad \Rightarrow \quad \log K = \dfrac{x}{20} \quad \Rightarrow \quad \therefore K = 10^{\frac{x}{20}}$

EXAMPLE 3.13

Consider a unity feedback system having an open loop transfer function $G(s) = \dfrac{K(1+10s)}{s^2\,(1+s)\,(1+2s)}$. Sketch the Nichols plot and determine the value of K so that (i) Gain margin is 10db, (ii) Phase margin is 10°.

SOLUTION

Given that $G(s) = \dfrac{K(1+10s)}{s^2(1+s)(1+2s)}$

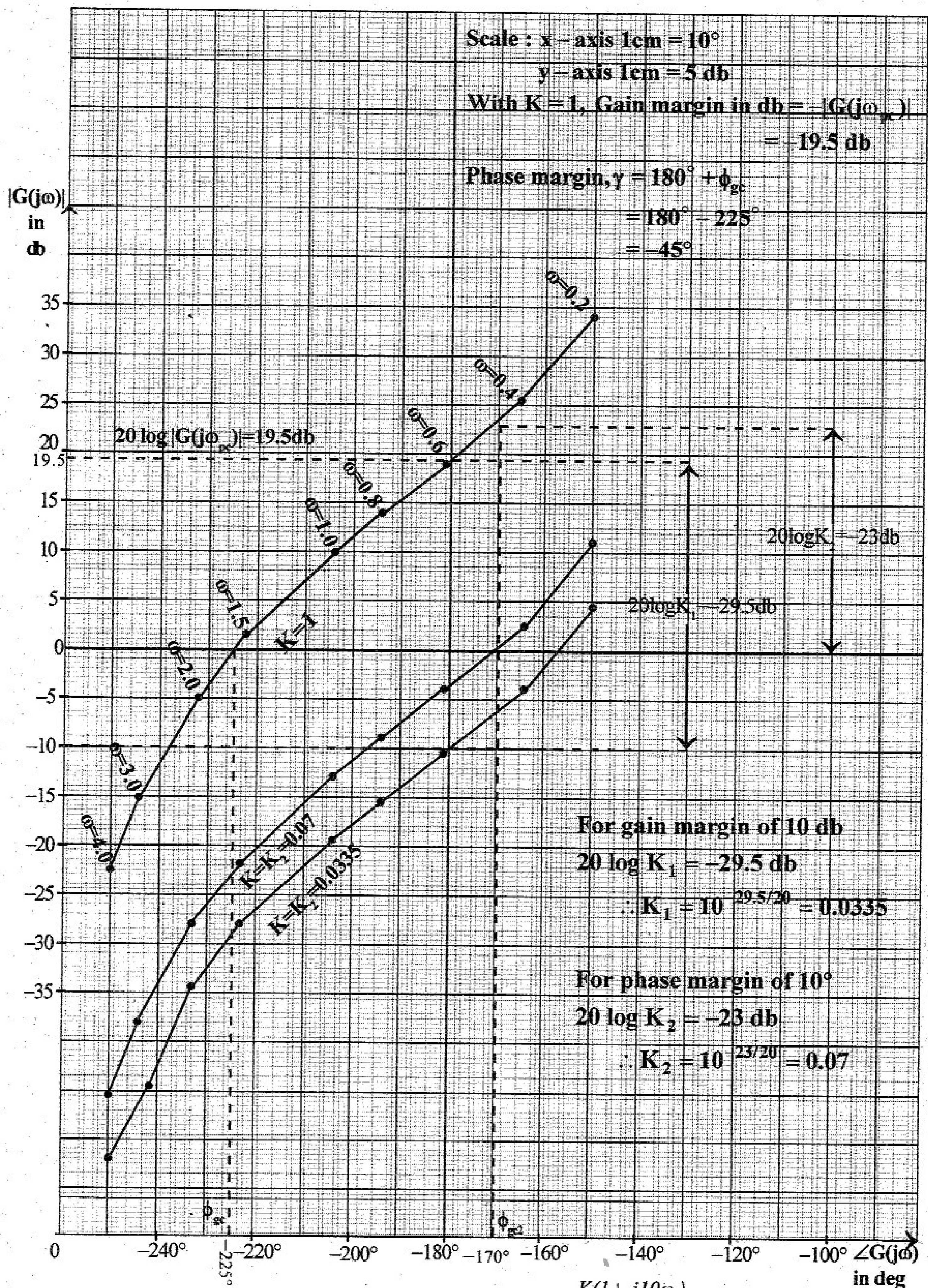


Fig 3.13.1 : Nichols plot of $G(j\omega) = \dfrac{K(1+j10\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)}$.

The sinusoidal transfer function G(jω) is obtained by letting to s = jω. Also put K = 1.

$$\therefore G(j\omega) = \frac{(1+j10\omega)}{(j\omega)^2(1+j\omega)(1+j2\omega)} = \frac{\sqrt{1+(10\omega)^2}\ \angle\tan^{-1}10\omega}{\omega^2\angle 180^0\sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\sqrt{1+(2\omega)^2}\ \angle\tan^{-1}2\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}; \qquad \therefore |G(j\omega)|_{\text{indb}} = 20\log\left[\frac{\sqrt{1+100\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\right]$$

$$\angle G(j\omega) = \tan^{-1}10\omega - 180^0 - \tan^{-1}\omega - \tan^{-1}2\omega$$

The magnitude of G(jω) in db and phase of G(jω) in deg are calculated for various values of ω and listed in the following table. The Nichols plot of G(jω) with K = 1 is sketched as shown in fig 3.13.1

ω rad/sec	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0
|G(jω)| db	34.1	25.4	19.3	14.3	10	1.4	–5.3	–15.2	–22.5
∠G(jω) deg	–150	–164	–181	–194	–204	–222	–232	–244	–250

From the Nichols plot the gain margin and phase margin of the system when K=1 are,

Gain margin = – 19.5 db

Phase margin = – 45°

Gain adjustment for required gain margin

For a gain margin of 10 db, the magnitude of G(jω) should be –10db, when the phase is –180°. When K = 1, the magnitude of G(jω) is +19.5db corresponding to phase angle of –180°. Hence if we add –29.5 db to every point of G(jω) ,then the plot shifts downwards and it will cross –180° axis at a magnitude of –10db. The magnitude correction is independent of frequency and so this gain can be contributed by the term K. Let this value of K be K_1. The value of K_1 is calculated by equating $20\log K_1$ to –29.5db.

$$\therefore 20\log K_1 = -29.5\text{ db} \quad\Rightarrow\quad \log K_1 = \frac{-29.5}{20} \quad\Rightarrow\quad K_1 = 10^{\frac{-29.5}{20}} = 0.0335$$

Gain adjustment for required phase margin

Let ϕ_{gc2} = phase of G(jω) at gain crossover frequency for a phase margin of 10°

$\therefore$ Phase margin, $\gamma_2 = 180° + \phi_{gc2}$

$\therefore \phi_{gc2} = \gamma_2 - 180° = 10° - 180° = -170°$

When K = 1, the magnitude of G(jω) is +23 db corresponding to a phase of –170°. But for a phase margin of 10°, this gain should be made zero. Hence if we add –23db to every point of G(jω) locus then the plot shifts downwards and it will cross –170° axis at magnitude of 0 db. The magnitude correction is independent of frequency and so this gain can be contributed by the term K. Let this value of K be K_2. The value of K_2 is calculated by equating $20\log K_2$ to –23db.

$$\therefore 20\log K_2 = -23 \quad\Rightarrow\quad \log K_2 = -23/20 \quad\Rightarrow\quad K_2 = 10^{\frac{-23}{20}} = 0.07$$

RESULT

(a) When K = 1,

Gain margin = –19.5 db

Phase margin = – 45°

(b) For a gain margin of 10db, $K = K_1$ = 0.0335

(c) For a phase margin of 10°, $K = K_2$ = 0.07

3.9 CLOSED LOOP RESPONSE FROM OPEN LOOP RESPONSE

The closed loop transfer function of the system is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s)$$

The sinusoidal transfer function is obtained by replacing s by jω.

$$\therefore\ M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)\,H(j\omega)}$$

Let, $M(j\omega) = M\angle\alpha$

where, M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

The magnitude and phase of closed loop system are functions of frequency, ω. The sketch of magnitude and phase of closed loop system with respect to ω is closed loop frequency response plot. The magnitude and phase of closed loop system for various values of frequency can be evaluated analytically or graphically. The analytical method of determining the frequency response involves tedious calculations. Two graphical methods are available to determine the closed loop frequency response from open loop frequency response. They are,

1. M and N circles
2. Nichols chart.

3.10 M AND N CIRCLES

The magnitude of closed loop transfer function with unity feedback can be shown to be in the form of circle for every value of M. These circles are called **M-circles.**

If the phase of closed loop transfer function with unity feedback is α, then it can be shown that tan α will be in the form of circle for every value of α. These circles are called **N-circles.**

The M and N circles are used to find the closed loop frequency response graphically from the open loop frequency response G(jω) without calculating the magnitude and phase of the closed loop transfer function at each frequency.

The M and N circles are available as standard chart. The chart consists of M and N circles superimposed on ordinary graph sheet. Using ordinary graph the locus of G(jω) (Polar Plot) is sketched. The locus of G(jω) will cut the M-circles and N-circles at various points. The intersection of G(jω) locus with M and N circles gives the magnitude and phase of the closed loop system at frequencies corresponding to the cutting point of G(jω).

The M and α for various values of ω are tabulated. The magnitude and phase response of closed loop system are sketched on semilog graph sheet by taking ω on the logarithmic scale on x-axis. [The closed loop frequency response has two plots. They are M Vs ω and α Vs ω]

M-CIRCLES

Consider the closed loop transfer function of unity feedback system, $M(s) = \dfrac{G(s)}{1+G(s)}$

Put $s = j\omega$, $\therefore\ M(j\omega) = \dfrac{G(j\omega)}{1+G(j\omega)}$

Let, $G(j\omega) = X + jY$

where, X = Real part of $G(j\omega)$.

Y = Imaginary part of $G(j\omega)$.

$$\therefore M(j\omega) = \frac{X+jY}{1+X+jY} = \frac{\sqrt{X^2+Y^2}\angle \tan^{-1}\dfrac{Y}{X}}{\sqrt{(1+X)^2+Y^2}\angle \tan^{-1}\dfrac{Y}{1+X}} = \frac{\sqrt{X^2+Y^2}}{\sqrt{(1+X)^2+Y^2}}\angle\left(\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y}{1+X}\right)$$

Let, M = Magnitude of $M(j\omega)$

$$\therefore M = \frac{\sqrt{X^2 + Y^2}}{\sqrt{(1+X)^2+Y^2}}$$

On squaring the above equation we get,

$$M^2 = \frac{X^2+Y^2}{(1+X)^2+Y^2} \quad \Rightarrow \quad M^2\left((1+X)^2+Y^2\right) = X^2 + Y^2 \quad \Rightarrow \quad M^2\left(1+X^2+2X+Y^2\right) = X^2 + Y^2$$

$$M^2 + M^2X^2 + M^22X + M^2Y^2 - X^2 - Y^2 = 0$$

$$X^2(M^2-1) + M^22X + M^2 + Y^2(M^2-1) = 0 \qquad .....(3.30)$$

When $M = 1$, the equation (3.30) represents a straight line.

When $M = 1$, the equation (3.30) is,

$$X^2(1-1) + 2X + 1 + Y^2(1-1) = 0 \quad \Rightarrow \quad 2X + 1 = 0 \quad \Rightarrow \quad X = -1/2$$

Hence when $M = 1$, equation (3.30) represents a straight line passing through $X = -1/2$ & $Y = 0$.

When $M \neq 1$, the equation (3.30) represents a family of circles.

When $M \neq 1$, equation (3.30) can be rearranged in the form of equation of a circle as shown below.

$$X^2(M^2-1) + M^22X + M^2 + Y^2(M^2-1) = 0$$

Divide the above equation throughout by $(M^2 - 1)$.

$$\therefore\ X^2 + \frac{M^2}{M^2-1}2X + \frac{M^2}{M^2-1} + Y^2 = 0$$

Add $\dfrac{M^2}{\left(M^2-1\right)^2}$ on both sides of the above equation.

$$X^2+\frac{M^2}{M^2-1}2X+\frac{M^2}{M^2-1}+\frac{M^2}{\left(M^2-1\right)^2}+Y^2=\frac{M^2}{\left(M^2-1\right)^2}$$

$$X^2+\frac{M^2}{M^2-1}2X+\frac{M^2\left(M^2-1\right)+M^2}{\left(M^2-1\right)^2}+Y^2=\frac{M^2}{\left(M^2-1\right)^2}$$

$$X^2+\frac{M^2}{M^2-1}2X+\frac{M^4}{\left(M^2-1\right)^2}+Y^2=\frac{M^2}{\left(M^2-1\right)^2}$$

$\boxed{a^2 + 2ab + b^2 = (a+b)^2}$

$$\left(X+\frac{M^2}{M^2-1}\right)^2+Y^2=\frac{M^2}{\left(M^2-1\right)^2} \qquad .....(3.31)$$

The equation of circle with centre at (X_1, Y_1) and radius r is given by,

$$(X-X_1)^2+(Y-Y_1)^2=r^2 \qquad .....(3.32)$$

On comparing equation (3.31) and equation (3.32), it can be concluded that the equation (3.31) represents a family circles with centre at $(-M^2/M^2-1), 0)$ and with radius, $r = M/(M^2-1)$ for various values of M. The circles given by equation (3.31) are called M-circles.

When M = 0

Centre $=(X_1,Y_1)$

$$X_1=-\frac{M^2}{M^2-1}=0$$

$$Y_1=0$$

$$\text{Radius, } r=\frac{M^2}{M^2-1}=0$$

Hence when M = 0, the magnitude circle becomes a point at (0,0).

When M = ∞

Centre $=(X_1,Y_1)$

$$X_1=\frac{-M^2}{M^2-1}\approx\frac{-M^2}{M^2}=-1$$

$$Y_1=0$$

$$\text{Radius, } r=\frac{M}{M^2-1}\approx\frac{M}{M^2}=\frac{1}{M}=\frac{1}{\infty}=0$$

Hence when $M=\infty$, the magnitude circle becomes a point at (−1,0).

From the above analysis it is clear that the magnitude of closed loop transfer function will be in the form of circles when $M \neq 1$ and when M = 1, the magnitude is a straight line passing through (−1/2,0).

For values of M less than 1, the magnitude is a circle to the right of the straight line corresponding to M = 1. It is observed that the circles for M<1 passes through (−1/2,0) and (0,0) on the negative real axis. For decreasing values of M, the radius decreases and the circle, becomes a point at (0,0) when M = 0.

For values of M greater than 1, the magnitude is a circle to the left of the straight line corresponding to M = 1. It is observed that circle passes between the points (−1,0) and (−1/2,0) on the negative real axis. For increasing values of M the radius decreases and the circle becomes a point at (−1,0) when $M=\infty$. The family of M-circles are shown in fig 3.28.

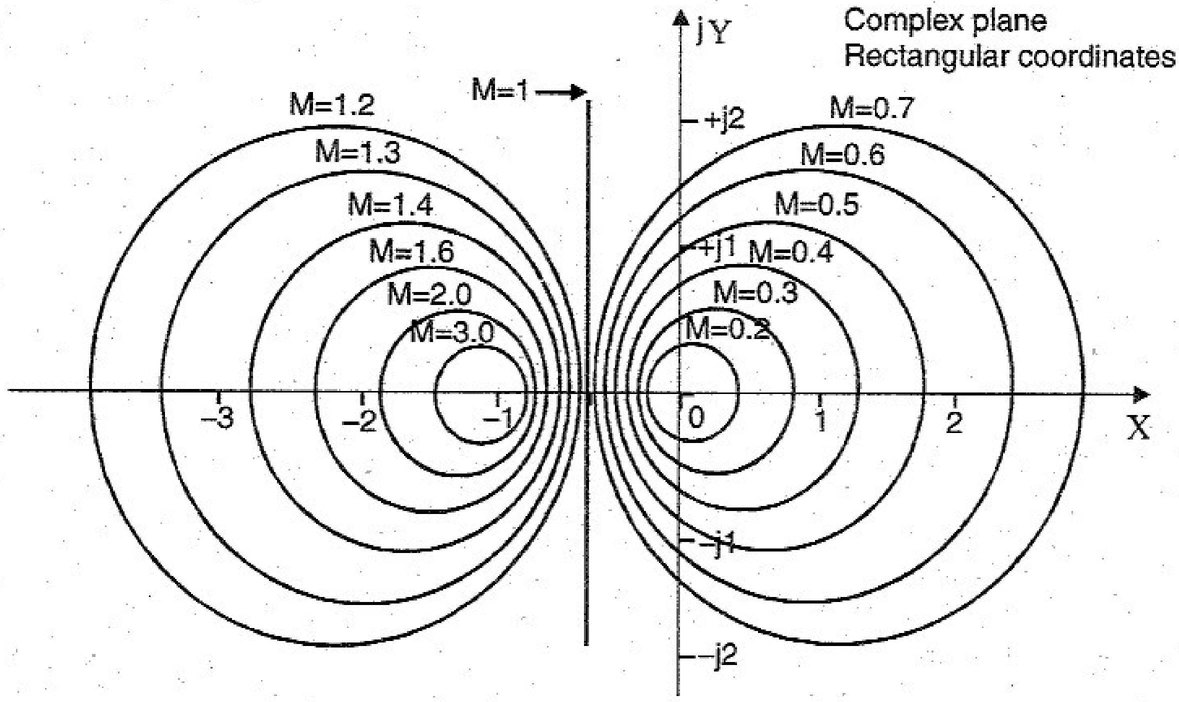


Fig 3.28 : The family of constant M-circles.

N-CIRCLES

Consider the closed loop transfer function of unity feedback system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = M(s)$$

Put $s = j\omega$, $M(j\omega) = \dfrac{G(j\omega)}{1+G(j\omega)}$

Let $G(j\omega) = X + jY$, where, X = Real part of $G(j\omega)$.

Y = Imaginary part of $G(j\omega)$.

$$\therefore M(j\omega) = \frac{X+jY}{1+X+jY} = \frac{\sqrt{X^2+Y^2}\angle \tan^{-1}\dfrac{Y}{X}}{\sqrt{(1+X)^2+Y^2}\angle \tan^{-1}\dfrac{Y}{1+X}} = \frac{\sqrt{X^2+Y^2}}{\sqrt{(1+X)^2+Y^2}}\angle\left(\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y}{1+X}\right)$$

Let, α = Phase of $M(j\omega)$; $\therefore \alpha = \tan^{-1}\dfrac{Y}{X} - \tan^{-1}\dfrac{Y}{1+X}$

Let, $N = \tan\alpha$

$$\therefore N = \tan\left(\tan^{-1}\frac{Y}{X} - \tan^{-1}\frac{Y}{1+X}\right)$$

$$\therefore N = \frac{\tan\left(\tan^{-1}\dfrac{Y}{X}\right) - \tan\left(\tan^{-1}\dfrac{Y}{1+X}\right)}{1+\tan\left(\tan^{-1}\dfrac{Y}{X}\right)\tan\left(\tan^{-1}\dfrac{Y}{1+X}\right)} = \frac{\dfrac{Y}{X} - \dfrac{Y}{1+X}}{1+\dfrac{Y}{X}\times\dfrac{Y}{1+X}} = \frac{\dfrac{Y(1+X)-XY}{X(1+X)}}{\dfrac{X(1+X)+Y^2}{X(1+X)}}$$

$$= \frac{Y+XY-XY}{X+X^2+Y^2} = \frac{Y}{X+X^2+Y^2}$$

$$\therefore N = \frac{Y}{X+X^2+Y^2}$$

> ***Note :***
>
> $$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \times \tan B}$$

On rearranging the above equation we get,

$$X + X^2 + Y^2 = \frac{Y}{N}$$

$$\therefore X + X^2 + Y^2 - \frac{Y}{N} = 0$$

In the above equation add the term $\frac{1}{4} + \left(\frac{1}{2N}\right)^2$ on both sides.

$$X + X^2 + Y^2 - \frac{Y}{N} + \frac{1}{4} + \left(\frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2$$

$$\left(X^2 + \frac{1}{4} + X\right) + \left(Y^2 + \frac{1}{(2N)^2} - \frac{Y}{N}\right) = \frac{1}{4} + \frac{1}{(2N)^2}$$

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2} \quad .....(3.33)$$

The equation of circle with centre at (X_1, Y_1) and radius r is,

$$(X - X_1)^2 + (Y - Y_1)^2 = r^2 \quad .....(3.34)$$

On comparing equation (3.33) and (3.34), it can be concluded that the equation (3.33) represents a family of circle with centre at (–1/2,1/2N) and with radius $\sqrt{\frac{1}{4} + \frac{1}{(2N)^2}}$ for various values of N. The circles given by the equation (3.30) are called N-circles.

For any value of N, the equation of N-circles is satisfied at two points (0,0) and (–1,0). Hence the N-circles passes through these two points for all values of α. ($N = \tan \alpha$).

Consider the equation of N-circle,

When X = 0 and Y = 0,

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2}$$

$$\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2}$$

$$\frac{1}{4} + \frac{1}{4N^2} = \frac{1}{4} + \frac{1}{4N^2}$$

Consider the equation of N-circle,

When X = –1 and Y = 0,

$$\left(X + \frac{1}{2}\right)^2 + \left(Y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2}$$

$$\left(-1 + \frac{1}{2}\right)^2 + \left(-\frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(4N)^2}$$

$$\frac{1}{4} + \frac{1}{4N^2} = \frac{1}{4} + \frac{1}{4N^2}$$

The above analysis shows that the equation of N-circle is satisfied at points(0,0) and (–1, 0).

When $\alpha = 180°$ the circle becomes a straight line passing through real axis. It is also observed that the circle for $\alpha = \theta° – 180°$ above the real axis will be a part of circle for $\alpha = \theta\,°$ below the real axis, as shown in fig 3.29. The family of N circles are shown in fig 3.30.

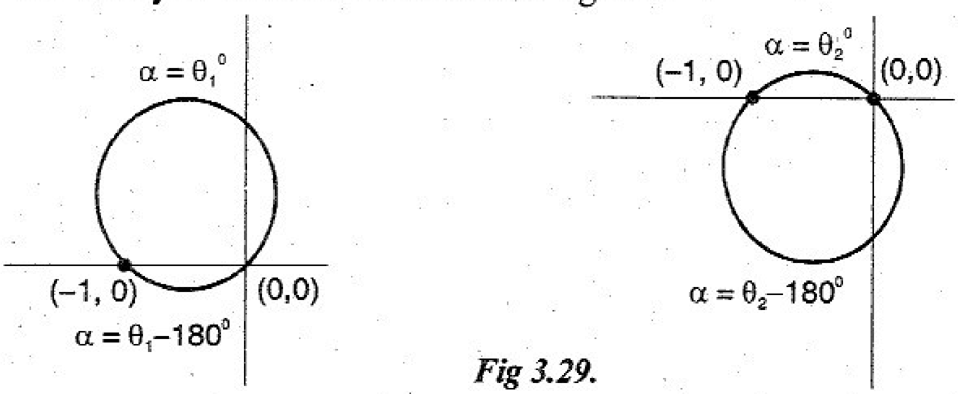


Fig 3.29.

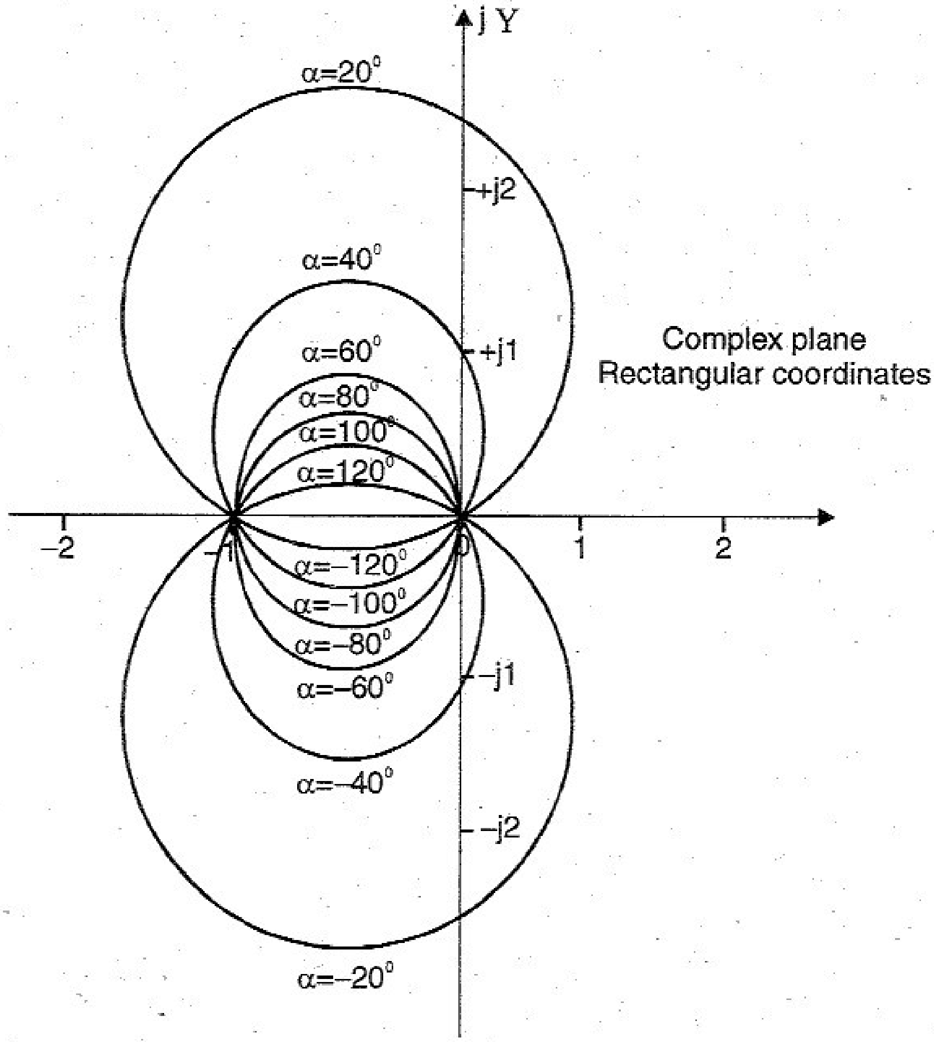


Fig 3.30 : The family of constant N-circles.

3.11 NICHOLS CHART

N.B Nichols transformed the constant M and N circles to log-magnitude and phase angle coordinates and the resulting chart is known as ***Nichols chart.***

Nichols chart consist of M and N contours, superimposed on ordinary graph. The M contours are the magnitude of closed loop system in decibels and the N contours are the corresponding phase angle locus of closed loop system. The ordinary graph consist of magnitude in db marked on the Y-axis and the phase in degrees marked on the X-axis.

The Nichols plot of open loop system can be plotted on the ordinary graph. The Nichols plot is a graph between magnitude of $G(j\omega)$ in db and the phase of $G(j\omega)$ in degree,plotted on an ordinary graph sheet. To draw the Nichols plot the magnitude and phase angle of $G(j\omega)$ are calculated for various values of ω. Alternatively the Bode plot of $G(j\omega)$ is sketched and from Bode plot, the magnitude and phase of $G(j\omega)$ for any frequency can be obtained.

Using Nichols chart the closed loop frequency response can be determined graphically from the locus of open loop frequency response. When the Nichols plot of $G(j\omega)$ is sketched on Nichols chart, the locus of $G(j\omega)$ will cut the M and N contours at various points. The cutting point of locus of $G(j\omega)$ with the M-contour gives the magnitude of closed loop frequency response corresponding to a frequency same as that of $G(j\omega)$ at that point.

The cutting point of locus of $G(j\omega)$ and N contour gives the phase of closed loop frequency response corresponding to a frequency same as that of $G(j\omega)$ at that point. The magnitude M and phase angle α ($N = \tan\alpha$) of closed loop system are tabulated . The closed loop frequency response consists of two plots. They are magnitude M Vs ω and phase angle α Vs ω. Hence using the tabulated values the bode plot of closed of closed loop system can be drawn.

The frequency domain specifications can be determined from Nicols chart. Fig 3.31, shows various frequency domain specifications of a typical $G(j\omega)$ locus. Also the Nichols plot drawn on a Nichols chart can be used for gain adjustment.

ESTIMATION OF FREQUENCY DOMAIN SPECIFICATIONS USING NICHOLS CHART

Resonant Peak (M_r) and Resonant Frequency (ω_r)

The resonant peak is given by the value of M-contour which is tangent to $G(j\omega)$ locus. The resonant frequency is given by the frequency of $G(j\omega)$ at the tangency point.

Bandwidth

The Bandwidth is given by frequency corresponding to the intersection point of $G(j\omega)$ and –3 db M -contour.

Gain Margin

The gain margin is given by negative of magnitude of $G(j\omega)$ in db at phase crossover frequency, ω_{pc}. At phase crossover frequency the phase of $G(j\omega)$ is $-180°$

Gain Margin, K_g in db $= -|G(j\omega_{pc})|_{in\ db}$

Phase Margin

The phase margin,γ is given by $\gamma = 180° + \phi_{gc}$ where ϕ_{gc} is the phase of $G(j\omega)$ at gain crossover frequency. At gain crossover frequency the magnitude of $G(j\omega)$ is zero db.

GAIN ADJUSTMENT USING NICHOLS CHART

Determination of K for Specified Gain Margin

Draw the $G(j\omega)$ locus with K=1. Determine the amount of gain to be added at $\phi = -180°$, so that db magnitude of $G(j\omega)$ locus at $-180°$ is negative of the specified gain margin. Let the db gain to be added be x db. The gain contribution is independent of frequency and so it can be achieved by choosing proper value of K. The value of K is obtained by equating 20logK to x db.

Now, $20\log K = x$

$$\therefore K = 10^{\frac{x}{20}}$$

Determination of K for Specified Phase Margin

Draw the $G(j\omega)$ locus with K= 1. The phase margin, $\gamma = 180° + \phi_{gc}$ where ϕ_{gc}, is phase of $G(j\omega)$ at gain crossover frequency. $\therefore \phi_{gc} = \gamma - 180°$. For specified phase margin, calculate ϕ_{gc} and from the Nichols plot determine the db gain at ϕ_{gc} . Let this gain be y db. For the specified phase margin, this gain should be made zero. Hence –y db should be added to every point of $G(j\omega)$. This is achieved by choosing proper value of K. The value of K is obtained by equating 20logK to –y db.

$$\text{Now, } 20\log K = -y \quad \Rightarrow \quad \log K = -\frac{y}{20} \quad \Rightarrow \quad K = 10^{\frac{-y}{20}}$$

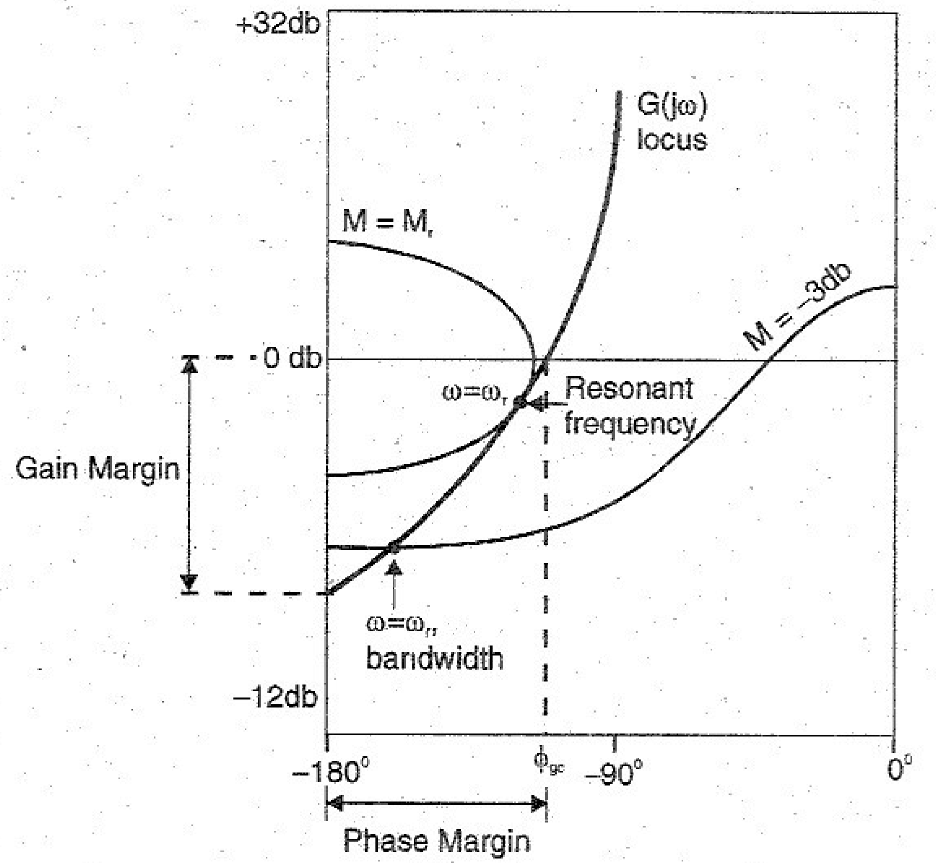


Fig 3.31 : Determination of frequency domain specification from Nichols Chart.

Determination of K for Specified Resonant Peak, M_r

Draw the G(jω) locus with K = 1. Using a tracing paper, trace the locus of G(jω). A standard tracing paper, called Nichols overlay is available). Then shift the locus vertically up or down, so that $M = M_r$ contour is tangent to G(jω) locus. Measure the vertical shift in db. Let the shift be ± x db. (+ for up and –for down).

$$\text{Now, } 20\log K = \pm x \quad \Rightarrow \quad \log K = \pm\frac{x}{20} \quad \Rightarrow \quad K = 10^{\pm\frac{x}{20}}$$

Determination of K for a Specified Bandwidth

Draw the G(jω) locus with K=1. Determine the open loop gain G(jω) at $\omega = \omega_b$ where, ω_b is the specified bandwith. Determine the point of intersection of -3db M-contour and this open loop gain on the Nichols chart. Let this point be point A. Trace the G(jω) locus. Shift the G(jω) locus vertically up or down, so that it passes through point A. Measure the vertical shift in db. Let the shift be ± x db (+for up and –for down).

$$\text{Now, } 20\log K = \pm x \quad \Rightarrow \quad \log K = \pm\frac{x}{20} \quad \Rightarrow \quad K = 10^{\pm\frac{x}{20}}$$

EXAMPLE 3.14

The open loop transfer function of unity feedback system is, $G(s) = Ke^{-0.2s}/s(1+0.25s)(1+0.1s)$. Using Nichols chart, determine the following.

(a) The value of K so that the gain margin of the system is 4 db.

(b) The value of K so that the phase margin of the system is 40°

(c) The value of K so that resonant peak M_r of the system is 1 db. What are the corresponding values of ω_r and ω_b?

(d) The value of K so that the bandwidth ω_b of the system is 1.5 rad/sec.

SOLUTION

First the actual bode plot of G(jω) with K=1 is plotted on semilog graph sheet. The magnitude of G(jω) in db and phase of G(jω) for various frequencies are calculated and listed in Table-1. The choice of frequencies are chosen such that the magnitude plot extends in the range of 40 db to –14db and the phase plot extends in the range of 0° to –180°.

Given that, $G(s) = \dfrac{Ke^{-0.2s}}{s(1+0.25s)(1+0.1s)}$

Let, K = 1 and put, s = jω.

$$\therefore G(j\omega) = \frac{e^{-j0.2\omega}}{j\omega(1+0.25s)(1+0.1s)} = \frac{1\angle -0.2\omega \times \frac{180^\circ}{\pi}}{\omega\angle 90^\circ \sqrt{1+0.0625\omega^2}\angle \tan^{-1}0.25\omega \sqrt{1+0.01\omega^2}\angle \tan^{-1}0.1\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+0.0625\omega^2}\sqrt{1+0.01\omega^2}}; \quad \therefore |G(j\omega)|_{\text{in db}} = 20\log\left[\frac{1}{\omega\sqrt{1+0.0625\omega^2}\sqrt{1+0.01\omega^2}}\right]$$

$$\angle G(j\omega) = -0.2\omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1}0.25\omega - \tan - 10.1\omega$$

TABLE-1: Calculated values of |G(jω)| and ∠G(jω)

ω rad/sec	0.01	0.02	0.05	0.1	0.2	0.5	1.0	2.0	4.0
|G(jω)| db	40	34	26	20	14	6	0	–7	–16
∠G(jω) deg	–90	–91	–91	–93	–96	–106	–121	–151	–203

The magnitude and phase plot of Bode plot of G(jω) are shown in fig 3.14.1 From the bode plot the phase and frequency for various values of magnitudes are noted and tabulated in table-2. (The choice of magnitudes are 20,16,12,.... i.e, in steps of 4 db, which is convenient for Nichols plot on Nichols chart). Using the values listed in table-2 the locus of G(jω) on Nichols chart is sketched as shown in fig 3.14.2.

TABLE-2: Values of |G(jω)| and ∠G(jω) Noted from Bode Plot

ω rad/sec	0.1	0.16	0.25	0.4	0.64	1.0	1.5	2.2	3.0
|G(jω)| db	20	16	12	8	4	0	– 4	– 8	–12
∠G(jω) deg	–90	–92	–96	–102	–110	–120	–136	–156	–180

Gain Margin and Phase Margin when K = 1

When K = 1, the G(jω) locus cuts the –180° axis at –12db. Hence the magnitude at phase crossover frequency is –12db.

$\therefore$Gain Margin, $K_g = -|G(j\omega_{pc})|_{\text{indb}} = -(-12) = +12\text{db}.$

When K = 1, the phase of G(jω) is –120° corresponding to magnitude of 0 db. Hence the phase at gain crossover frequency is –120°.

$\therefore$Phase margin, $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 120^\circ = 60^\circ$

To find K for a gain margin of 4db

When gain margin is 4 db, the locus of G(jω) should cross the 180° axis at –4 db. When K = 1, the magnitude of G(jω) is –12db corresponding to a phase of 180°. Hence if we add, –4 – (–12) = 8 db to every point of G(jω) then the plot shifts upwards and crosses –180° axis at –4 db. This magnitude correction is achieved by choosing appropriate value of K. The value of K is obtained by equating 20 log K to 8 db.

$\therefore 20\log K = 20\,\text{db} \quad \Rightarrow \quad \log K = \frac{8}{20} \quad \Rightarrow \quad K = 10^{\frac{8}{20}} = 2.5$

The G(jω), when K = 2.5 is shown in fig 3.14.2.

To find K for phase margin of 40°

Let ϕ_{gc2} be the phase of G(jω) at gain crossover frequency when the phase margin is 40°.

$\therefore$ Phase margin, $\gamma_2 = 180° + \phi_{gc2}$

$\therefore \phi_{gc2} = \gamma_2 - 180° = 40° - 180° = -140°$

From the above calculation it is evident that for a phase margin of 40°, the magnitude of G(jω) should be 0 db corresponding to a phase of –140°. When K = 1, the magnitude of G(jω) is –5db corresponding to a phase of –140°. Hence if we add +5 db to every point of G(jω) locus then the plot shifts upwards and crosses –140° axis at 0 db. This magnitude correction is achieved by choosing appropriate values of K. The value of K is obtained by equating 20 logK to 5 db.

$\therefore 20\log K = 5 \quad \Rightarrow \quad \log K = \frac{5}{20} \quad \Rightarrow \quad K = 10^{\frac{5}{20}} = 1.78$

The G(jω) locus, when K = 1.78 is shown in fig 3.14.2

To find K for a resonant peak of 1 db

The resonant peak, M_r is given by M-contour which is tangent to G(jω) locus. When K = 1, the G(jω) locus is tangent to M = 0.25 db contour. Hence when, K = 1, resonant peak is 0.25 db.

For a resonant peak of 1 db, the M = 1 db contour should be made tangent to G(jω) locus. For this, G(jω) locus can be shifted vertically up or down so that it becomes tangent to M = 1 db contour. In this problem the G(jω) locus is shifted vertically up to make it tangent to M = 1 db contour. The shifted G(jω) locus is shown in fig 3.14.3.

> *Note : Trace the G(jω) locus when K = 1 on a tracing paper and shift the traced locus over the Nichols chart vertically so that it is tangent to required M-contour. By keeping the tracing paper at the shifted position darken the traced locus, so that it makes an impression on nichols chart.*

The vertical shift is equivalent to adding a magnitude of 20 log K to every point of G(jω) locus. From the shifted locus of G(jω) it is observed that +2db is added to every point of G(jω) locus. Hence the value of K is obtained by equating 20 log K to +2 db.

$20\log K = 2\,\text{db} \quad \Rightarrow \quad \log K = \frac{2}{20} \quad \Rightarrow \quad K = 10^{\frac{2}{20}} = 1.26$

The resonant frequency, ω_r is given by the frequency of G(jω) at the tangency point. The magnitude of G(jω) is 0 db at the tangency point of M = 1 db contour. The corresponding frequency is noted from the bode plot of G(jω). From the bode plot the frequency at 0 db is 1.0 rad/sec. Hence the resonant frequency, ω_r = 1.0 rad/sec.

To find K so that ω_b = 1.5 rad/sec

The bandwidth, ω_b is given by the frequency of G(jω) corresponding to the meeting point of G(jω) locus and M = –3db contour. From the bode plot find the magnitude of G(jω) when ω = 1.5 rad/sec. From fig 3.14.1 it is observed that magnitude of G(jω) is – 4db when ω = 1.5 rad/sec.

In the Nichols chart, find the point where the M = –3db contour passes through – 4db line. Let this point be P. Now the G(jω) locus with K = 1 is shifted vertically down so that is passes through point P. The shifted G(jω) locus is shown in fig 3.1.3.

> *Note : Trace the G(jω) locus when K = 1 on a tracing paper and shift the raced locus over Nichols chart so that is passes through point P. By keeping the tracing paper at the shifted position, darken the traced locus, so that is makes an impression on Nichols chart.*

The vertical shift is equivalent to adding a magnitude of 20logK to every point of G(jω) locus. From the shifted locus of G(jω) it is observed that -6db is added to every point of G(jω) locus. Hence the value of K is obtained by equating 20logK to –6 db.

$20\log K = -6\,\text{db} \quad \Rightarrow \quad \log K = -\frac{6}{20} \quad \Rightarrow \quad K = 10^{-6/20} = 0.5$

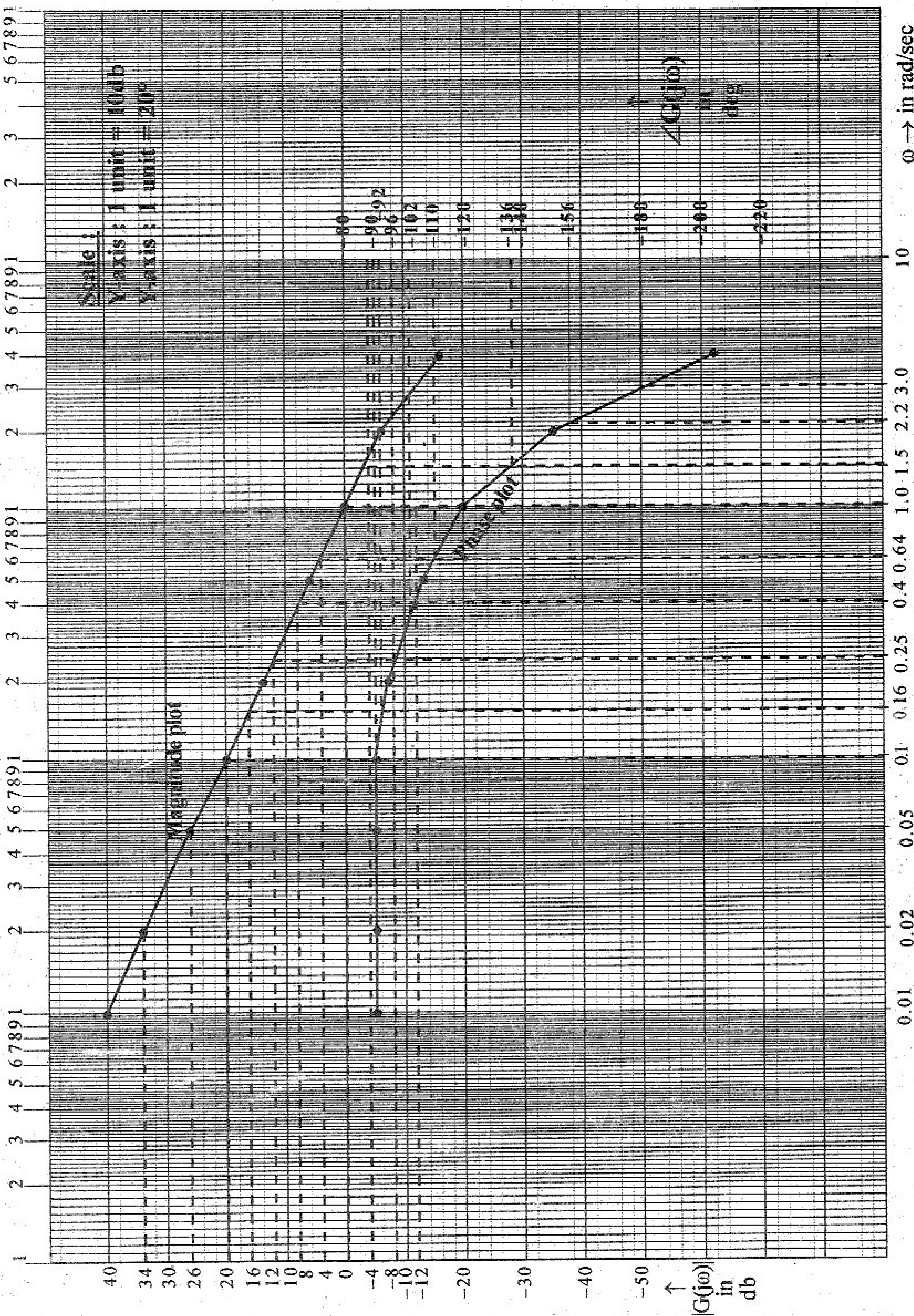


Fig 3.14.1 : Bode plot of $G(j\omega) = e^{-j0.2\omega}/j\omega\,(1 + j0.25\omega)\,(1 + j0.1\omega)$

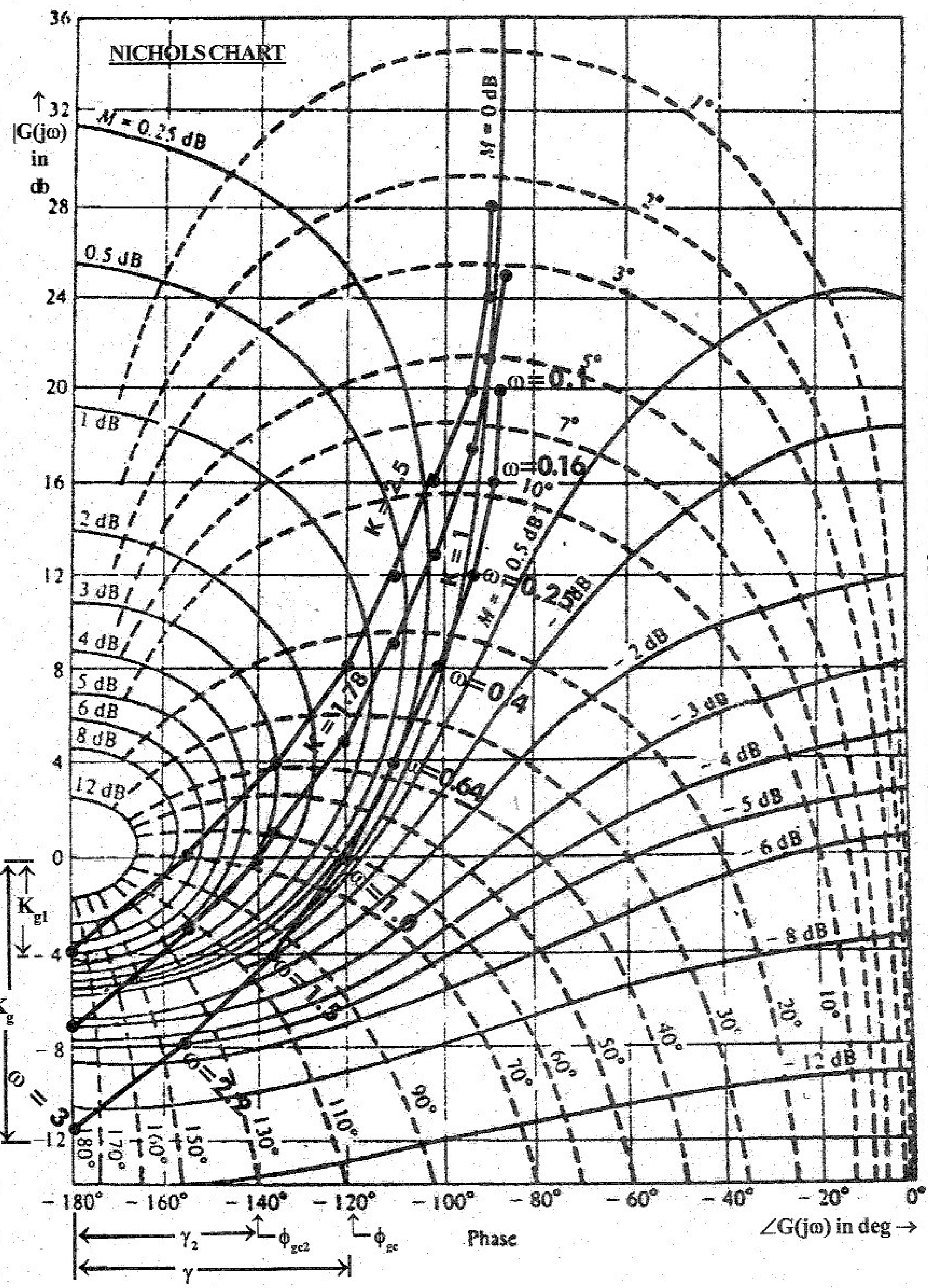


Fig 3.14.2 : Nichols plot of $G(j\omega) = Ke^{-j0.2\omega}/j\omega(1 + j0.25\omega)(1 + j0.1\omega)$

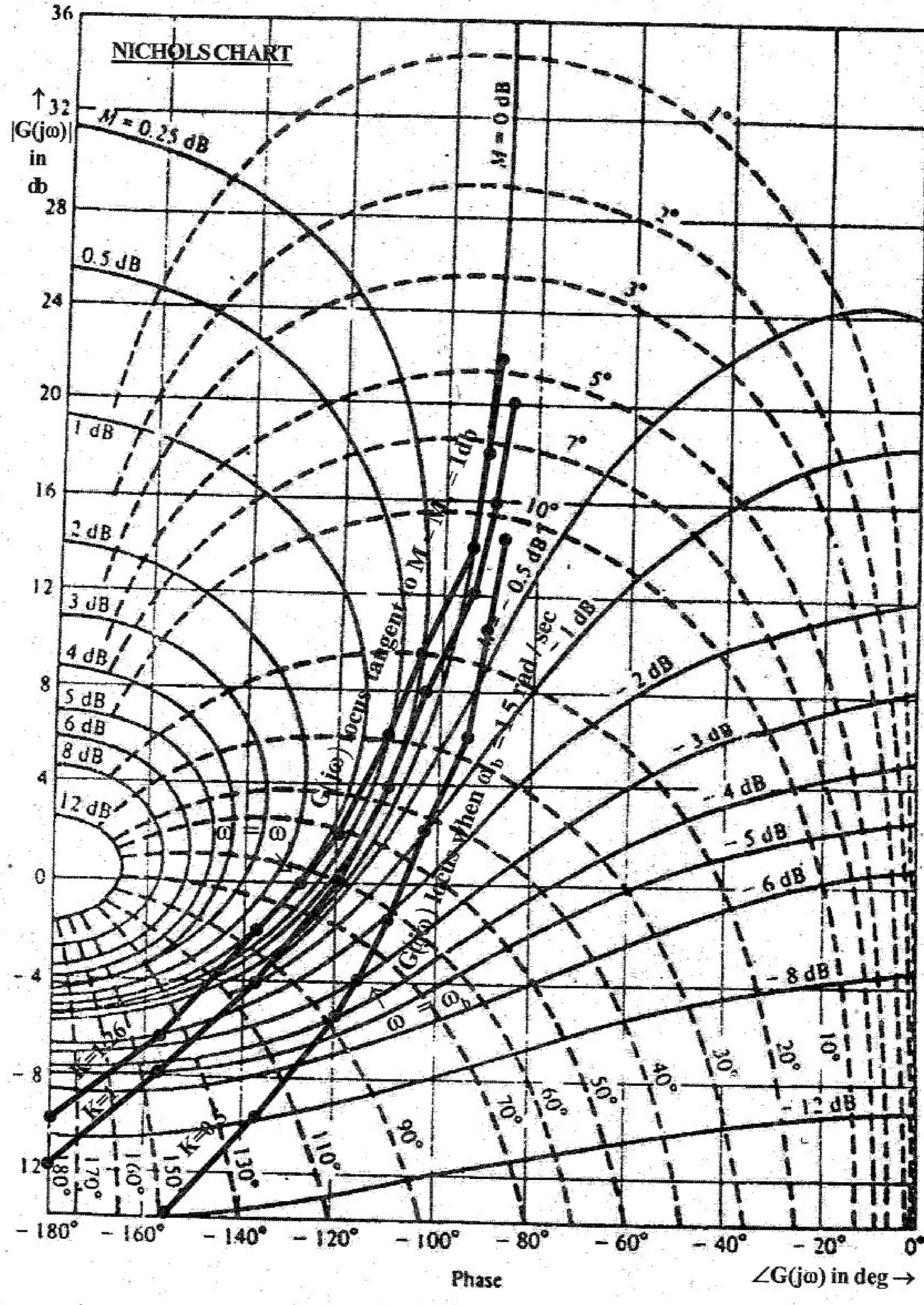


Fig 3.14.3 : Nichols plot of $G(j\omega) = Ke^{-j0.2\omega}/j\omega(1 + j0.25\omega)(1 + j\,0.1\omega)$

EXAMPLE 3.15

A unity feedback system has open loop transfer function, $G(s) = \frac{20}{s(s+2)(s+5)}$. Using Nichols chart, determine the closed loop frequency response and estimate M_r, ω_r and ω_b.

SOLUTION

Given that,

$$G(s) = \frac{20}{s(s+2)(s+5)}$$

The transfer function G(s) is converted to time constant or bode form.

$$G(s) = \frac{20}{s \times 2\left(\frac{s}{2}+1\right) \times 5\left(\frac{s}{5}+1\right)} = \frac{20/(2 \times 5)}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)} = \frac{2}{s\,(1+0.5s)\,(1+0.2s)}$$

Put, $s = j\omega$, in G(s) to get $G(j\omega)$.

$$\therefore G(j\omega) = \frac{2}{j\omega(1+j0.5\omega)(1+j0.2\omega)}$$

$$= \frac{2}{\omega\angle 90^\circ \sqrt{1+0.25\omega^2}\angle \tan^{-1}0.5\omega \sqrt{1+0.04\omega^2}\angle \tan^{-1}0.2\omega}$$

$$= \frac{2}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}}\angle(-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega)$$

$$\therefore |G(j\omega)| = \frac{2}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}}$$

$$|G(j\omega)|_{\text{in db}} = 20\log\left[\frac{2}{\omega\sqrt{1+0.25\omega^2}\sqrt{1+0.04\omega^2}}\right]$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}0.2\omega$$

The magnitude of $G(j\omega)$ in db and phase of $G(j\omega)$ for various frequencies are calculated and listed in table-1. The choice of frequencies are chosen such that the magnitude plot extends in the range of 40 db to –14 db and the phase plot extends in the range of 0° to –180°.

Using table-1, the actual bode plot of $G(j\omega)$ is plotted on semilog graph sheet, as shown in fig 3.15.1.

TABLE-1: Calculated values of $|G(j\omega)|$ and $\angle G(j\omega)$

ω, rad/sec	0.2	0.5	1.0	2.0	3.0	4.0
$\|G(j\omega)\|$, db	20	12	5	–4	–10	–15
$\angle G(j\omega)$, deg	–98	–110	–128	–157	–177	–192

The magnitude and phase plot of bode plot of $G(j\omega)$ are shown in fig 3.15.1 From the bode plot, the phase and frequency for various values of magnitudes are noted and tabulated in table-2. (The choice of magnitudes are 20, 16, 12,..., i.e., in steps of 4 db, which is convenient for Nichols plot on Nichols chart).

Using the values listed in table-2, the locus of $G(j\omega)$ is sketched on the Nichols chart as shown in fig 3.15.2.

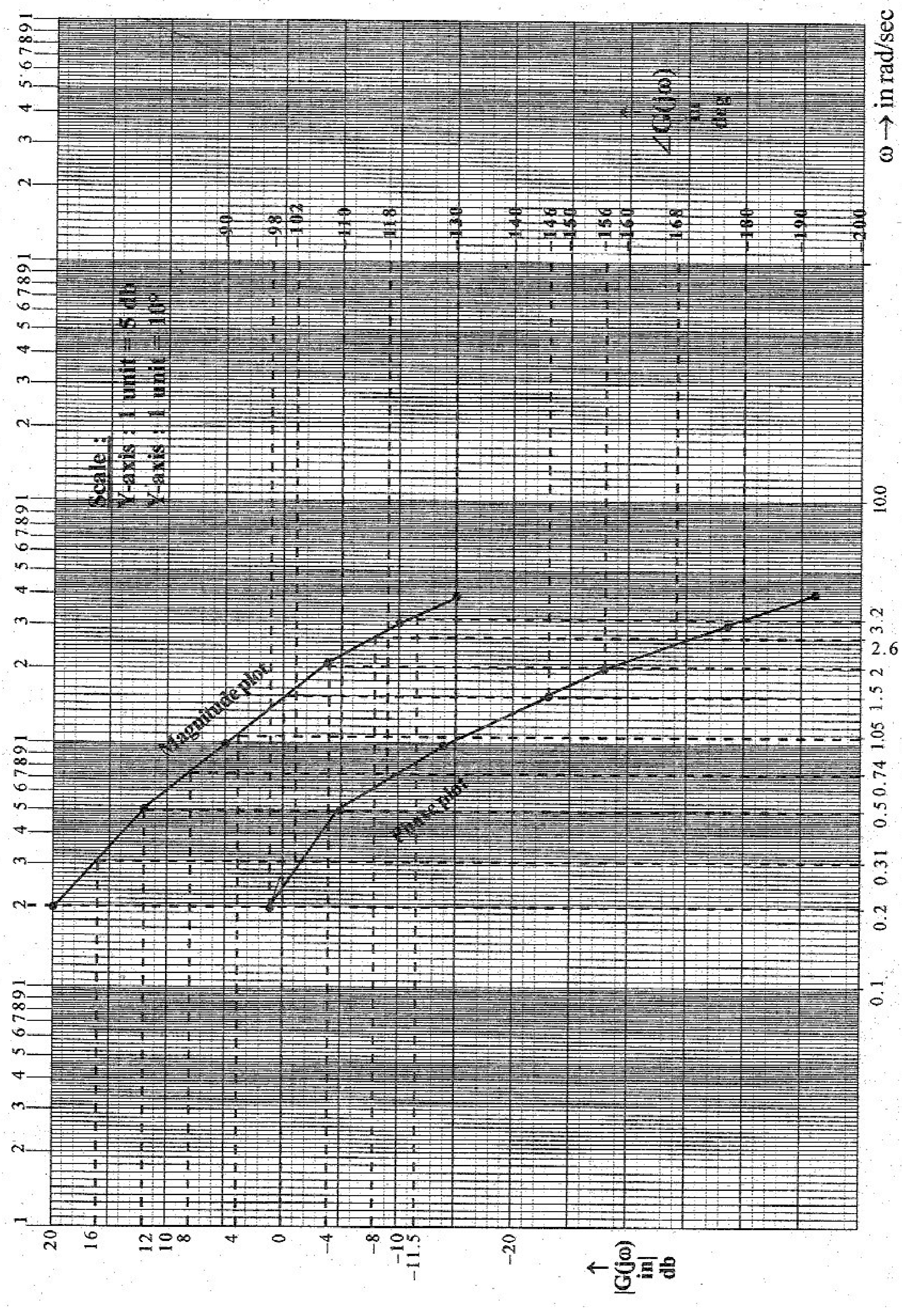


Fig 3.15.1 : Bode plot of $G(j\omega) = 2/[j\omega\,(1 + j0.5\,\omega)\,(1 + j0.2\,\omega)]$

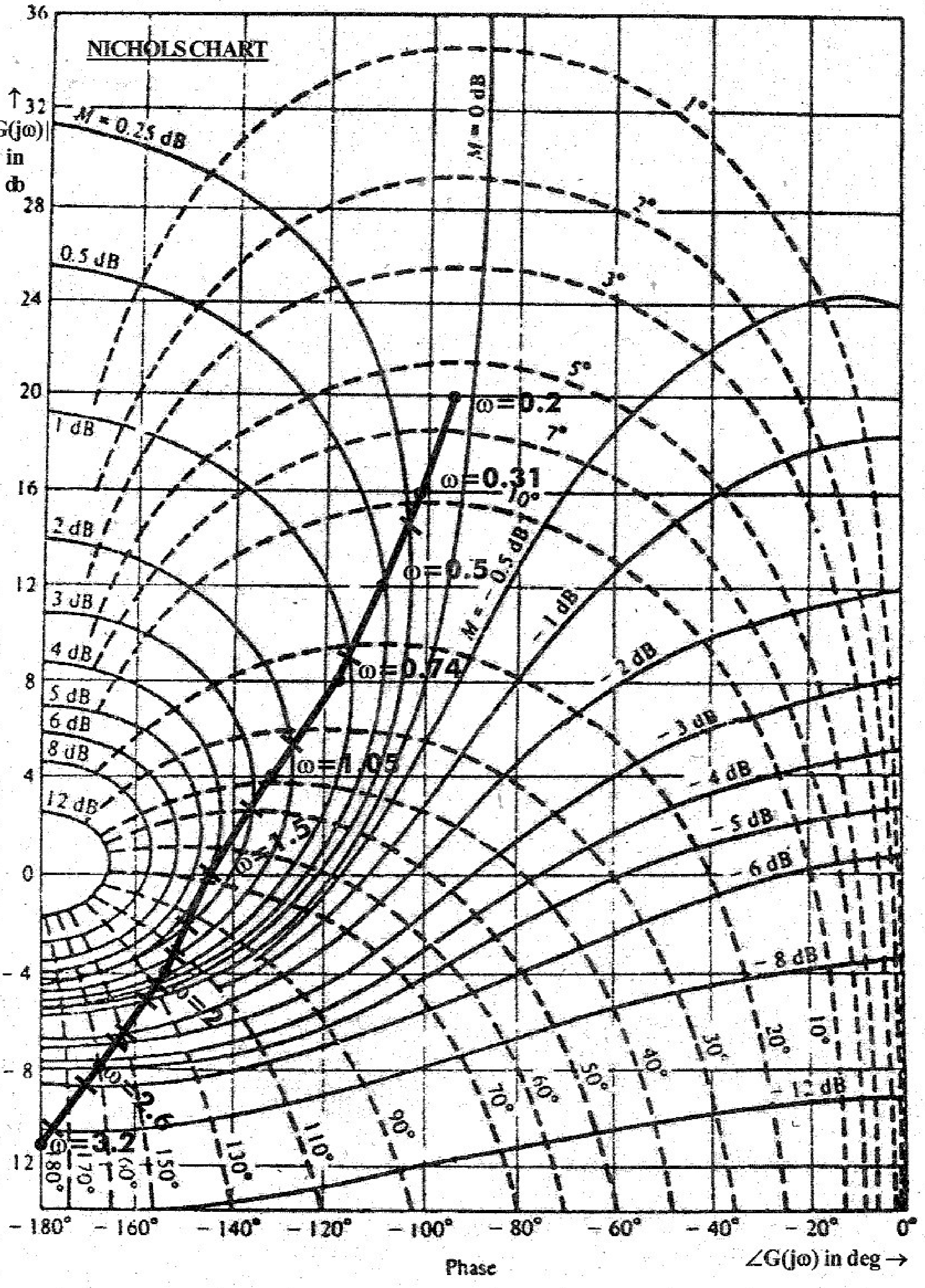


Fig 3.15.2 : Nichols plot of $G(j\omega) = 2/[j\omega\,(1 + j0.5\omega)\,(1 + j\,0.2\omega)]$.

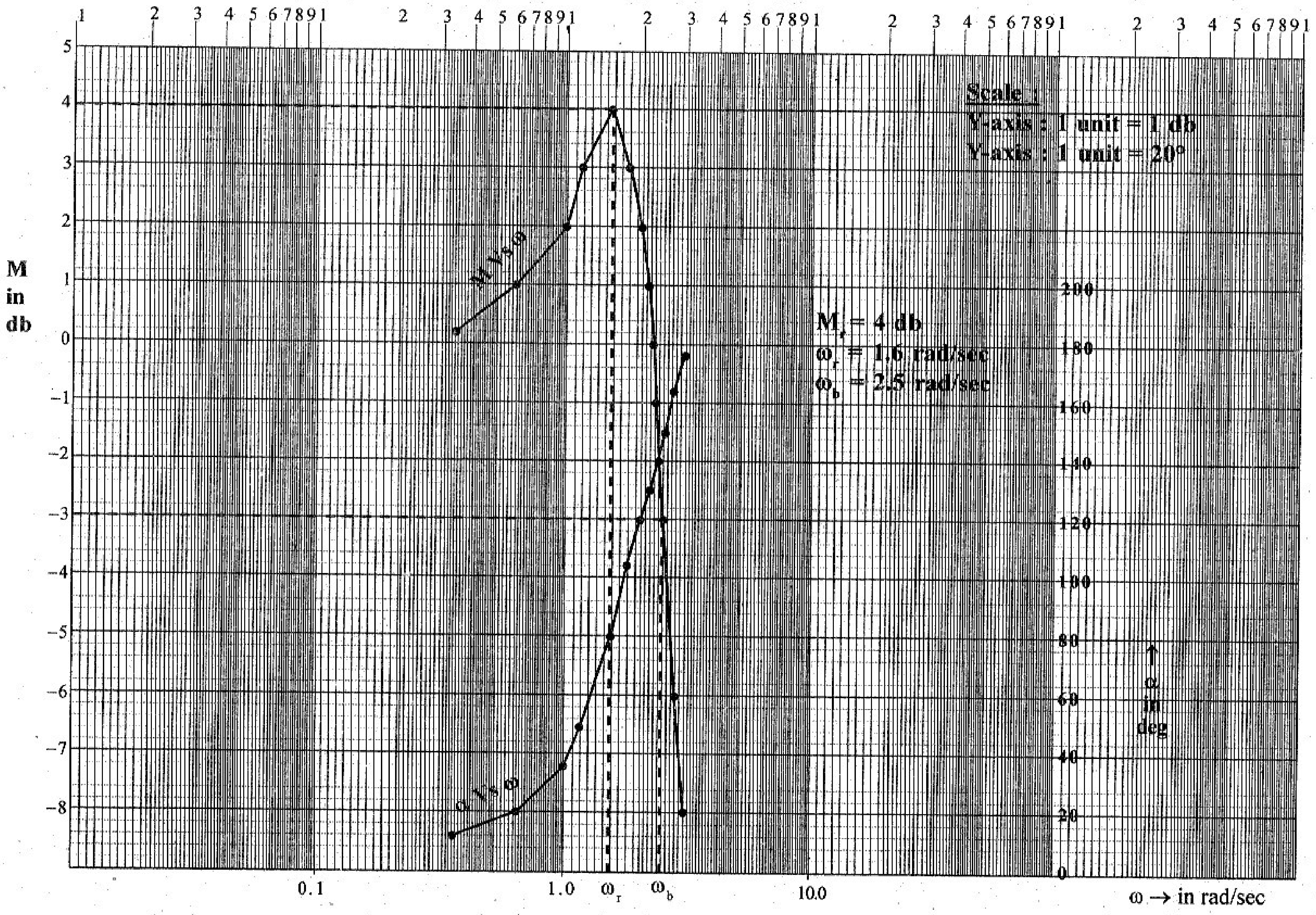


Fig 3.15.3 : Closed loop frequency response of $G(j\omega) = 2/[j\omega\,(1 + j0.5\omega)\,(1 + j0.2\omega)]$

TABLE-2: Values of $|G(j\omega)|$ and $\angle G(j\omega)$ Obtained from Bode Plot

ω rad/sec	0.2	0.31	0.5	0.74	1.05	1.5	2.0	2.6	3.2
$\|G(j\omega)\|$ db	20	16	12	8	4	0	−4	−8	−11.5
$\angle G(j\omega)$ deg	−98	−102	−110	−118	−130	−146	−156	−168	−180

The locus of $G(j\omega)$ drawn on the Nichols chart cuts the M-contour and N-contour at various points. The meeting points of $G(j\omega)$ locus and various M-contours are noted.

The phase α corresponding to the meeting point are noted from N-contours passing through the meeting point. (If a meeting point lies between two N-contours, then choose an approximate value of α).

The frequency corresponding to the meeting point are noted from bode plot by transferring the $|G(j\omega)|$ corresponding to meeting point to bode plot. The values of ω, M and α are listed in table-3.

The values of M and α are the magnitude and phase of closed loop frequency response of $G(j\omega)$ with unity feedback.

TABLE-3: Values of M and α from Nichols Chart

ω rad/sec	0.36	0.62	1.0	1.2	1.6	1.8	2.0	2.1	2.3	2.4	2.5	2.8	3.0
M db	0.25	1	2	3	4	3	2	1	0	−2	−3	−6	−8
α deg	12	21	35	50	78	102	120	130	140	151	155	165	175

Using the values listed in table-3, the closed loop frequency response plots are sketched as shown in fig 3.15.3. The closed loop frequency response consists of two plots and they are magnitude plot, M Vs ω and phase plot, α Vs ω.

From the closed loop frequency response the values of M_r, ω_r and ω_b are noted.

Resonant peak, M_r = +4 db

Resonant Frequency ω_r = 1.6 rad/sec

Bandwidth, ω_b = 2.5 rad/sec

3.12 FREQUENCY RESPONSE ANALYSIS USING MATLAB

In general, the open loop or closed loop transfer function of a system is denoted as T(s).

Let, T(s) be a rational function of "s", as shown below.

$$T(s) = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + \dots\dots\dots + b_{M-1}s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \dots\dots\dots + a_{N-1}s + a_N}$$

For frequency response analysis, the transfer function T(s) is declared as a function of s using the following commands.

```
s=tf('s');
Ts=(b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
```

The coefficients of numerator and denominator polynomials of the transfer function are determined using the following command.

```
[num_cof den_cof]=tfdata(Ts);
```

The gain margin, phase margin, gain crossover frequency and phase crossover frequency can be determined using the following command.

```
[GM PM wgc wpc] = margin(Ts);
```

BODE PLOT

In order to draw Bode plot the frequecy range can be specified using following commands.

```
w = logspace(ds, de, n);
where, ds represents Start decade as 10^ds
       de represents end decade as 10^de
       n represents number of points to be calculated between 10^ds & 10^de
```

Method 1 :

The Bode plot can be plotted using any one of the following command.

```
bode(Ts,'k');
bode(Ts, w);
bode(num_cof, den_cof);
bode(num_cof, den_cof,w);
```

Method 2 :

The Bode plot can also be plotted using semilog plot command as shown below.

```
[Mag Phase w] = bode(Ts,w);
MagdB = 20*log10(Mag);
subplot(2,1,1);semilogx(w,MagdB,'k');
subplot(2,1,2);semilogx(w,Phase,'k');
```

In this method the magnitude and phase can be scaled by drawing two lines at two specified upper and lower values. For drawing these lines, one dimensional arrays consisting of same values has to be created by multiplying the specified value with one. The length of the array should be same as number of frequency points for which the magnitude and phase are computed. (Refer program 3.3).

POLAR PLOT

The polar plot can be plotted using the following commands.

```
w = w_start : w_step : w_end ;
[re, im, w] = nyquist(num_cof,den_cof,w);
z = re + i*im; r = abs(z); theta = angle(z);
polar(theta,r,'w')
```

NICHOLS PLOT

The Nichols plot of open loop transfer function, G(s) can be plotted using the following commands.

```
[num_cof den_cof ] = tfdata(Gs);
nichols(Gs);
axis([ph_start, ph_end, mag_start, mag_end]);
```

PROGRAM 3.1

Write a MATLAB program to draw the Bode plot for the open loop system governed by the following transfer function.

$$G(s)=s^2/(1+0.2s)(1+0.02s)$$

```
%program to plot Bode plot

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=(s^2)/((1+0.2*s)*(1+0.02*s))

w=logspace(-1,2,200);          %specify the frequency range
bode(Gs,w)
grid
```

OUTPUT

```
The given transfer function is,

Transfer function:
                    s^2
          ----------------------
          0.004 s^2 + 0.22 s + 1
```

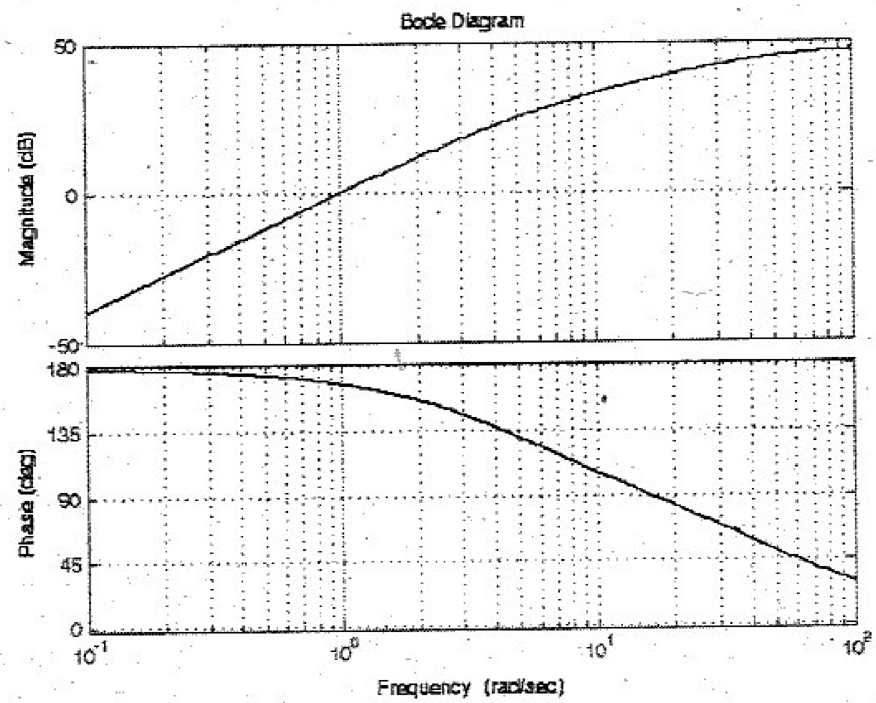


Fig P3.1 : Bode plot of the open loop system given in problem 3.1.

The Bode plot of program 3.1 is shown in fig p3.1.

PROGRAM 3.2

Write a MATLAB program to draw the Bode plot and to calculate gain margin, phase margin, gain crossover frequency & phase crossover frequency for the open loop system governed by the following transfer function.

$$G(s)=10/(0.04s^3+0.5s^2+s)$$

```
%program to find gain & phase margins using bode plot

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=10/((0.04*s^3)+(0.5*s^2)+s)

bode(Gs,'k')
grid

[GM,PM,wgc,wpc]=margin(Gs);
GMdB=20*log10(GM);
disp('Gain margin in dB,'); GMdB
disp('Phase margin in deg,');PM
disp('Gain cross over frequency in rad/sec,');wgc
disp('Phase cross over frequency in rad/sec,');wpc
```

OUTPUT

```
The given transfer function is,
Transfer function:
                    10
         ----------------------
         0.04 s^3 + 0.5 s^2 + s

Gain margin in dB,
          GMdB =
              1.9382

Phase margin in deg,
          PM =
              5.2057

Gain cross over frequency in rad/sec,
                         wgc =
                             5.0000
```

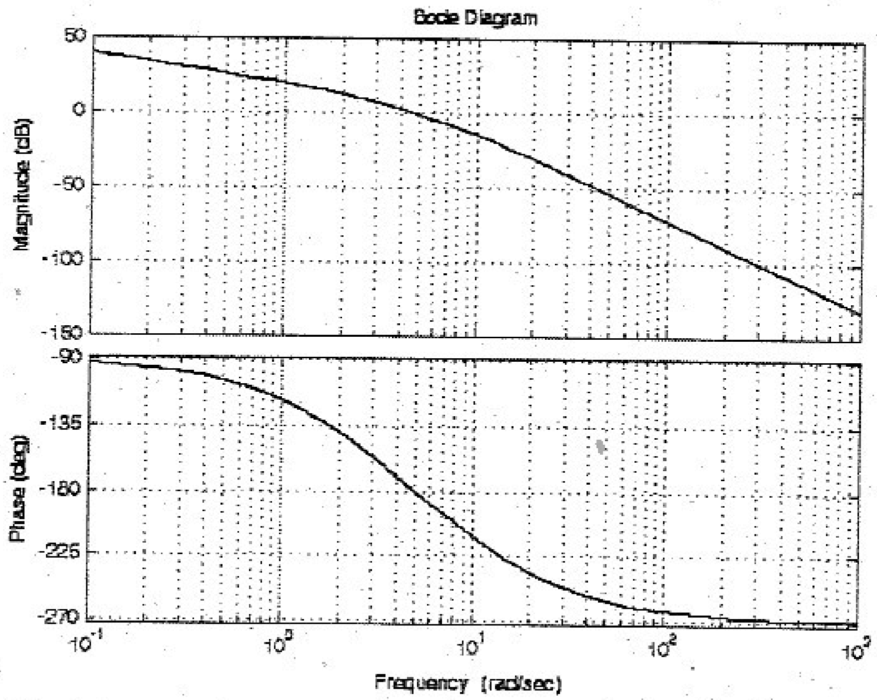


Fig P3.2 : Bode plot of the open loop system given in problem 3.2.

Phase cross over frequency in rad/sec,
wpc =
4.4629

The Bode plot of program 3.2 is shown in fig p3.2.

PROGRAM 3.3

Write a MATLAB program to draw the Bode plot for the open loop system governed by the following transfer function. The program should take care of drawing magnitude plot in the range +10 to -50 dB, and phase plot in the range -60 to -180 deg.

$G(s)=(75(1+0.2s))/(s(s^2+16s+100))$.

```
%Bode plot with magnitude and phase scaling

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=(75*(1+0.2*s))/(s*(s^2+16*s+100))
[num_cof den_cof]=tfdata(Gs); %determine numerator and denominator
                              %coefficients of G(s)

w=logspace(-1,2,200);
[Mag,Phase,w]=bode(num_cof,den_cof,w);
MagdB=20*log10(Mag);

Mscale1=10*ones(1,200);
Mscale2=-50*ones(1,200);
subplot(2,1,1);semilogx(w,MagdB,'-k',w,Mscale1,'k',w,Mscale2,'k')
grid;
xlabel('Frequency in rad/sec'); ylabel('Magnitude in dB');
Pscale1=-60*ones(1,200);
Pscale2=-180*ones(1,200);
subplot(2,1,2);semilogx(w,Phase,'k',w,Pscale1,'-k',w,Pscale2,'-k')
grid;
xlabel('Frequency in rad/sec'); ylabel('Phase in deg');
```

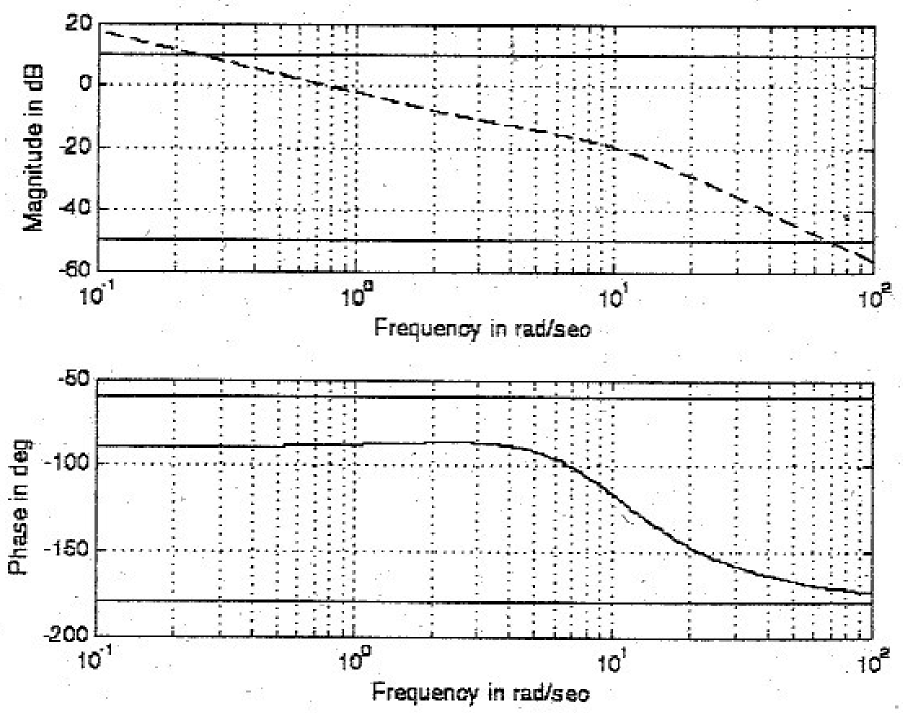


Fig P3.3 : Bode plot of the open loop system given in problem 3.3.

OUTPUT

```
The given transfer function is,

Transfer function:
                              15 s + 75
                         --------------------
                         s^3 + 16 s^2 + 100 s

The Bode plot of program 3.3 is shown in fig p3.3.
```

PROGRAM 3.4

Consider the transfer function of open loop system given below.

$G(s)=20/(s^3+7s^2+10s)$

Write a MATLAB program to determine the transfer function of closed loop system with unity feedback, to plot the Bode plot of closed loop system, and to calculate resonant peak, resonant frequency and bandwidth.

```
%Bode plot of unity feedback closed loop system

clear all
clc
s=tf('s');
disp('The given open loop transfer function G(S) is,');
Gs=20/(s^3+(7*s^2)+10*s)

disp('The closed loop transfer function M(s) is,');
Ms=feedback(Gs,1)
[num_cof den_cof]=tfdata(Ms); %determine numerator & denominator
                               %coefficients of M(s)
w=logspace(-1,1);              %specify frequency range
bode(Ms,w)
grid

[Mag,Phase,w]=bode(Ms,w);
[PeakMag,k]=max(Mag);
disp('Resonant peak in dB,');
Mp=20*log10(PeakMag)

disp('Resonant frequency in rad/sec,');Wr=w(k)

n=1; while 20*log(Mag(n))>=-3;n=n+1; end
disp('Bandwidth in rad/sec,');Wb=w(n)
```

OUTPUT

```
The given open loop transfer function G(S) is,

Transfer function:
                           20
                   ------------------
                   s^3 + 7 s^2 + 10 s

The closed loop transfer function M(s) is,

Transfer function:
                           20
                 -----------------------
                 s^3 + 7 s^2 + 10 s + 20
```

```
Resonant peak in dB,
                    Mp =
                         4.3953

Resonant frequency in rad/sec,
                    Wr =
                         1.6768

Bandwidth in rad/sec,
                    Wb =
                         2.4421
```

The Bode plot of program 3.4 is shown in fig p3.4.

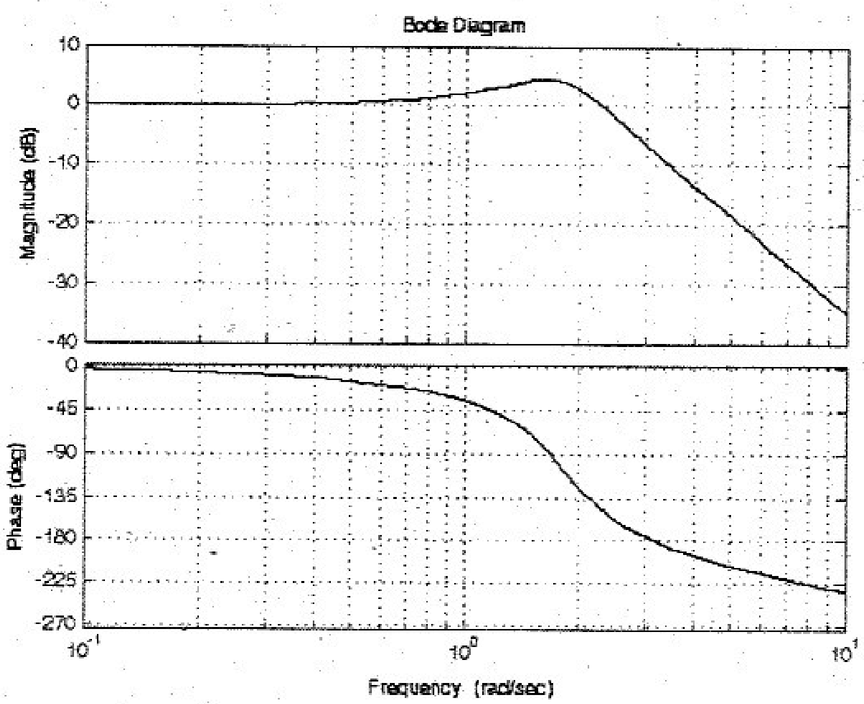


Fig P3.4 : Bode plot of the closed loop system of program 3.4.

PROGRAM 3.5

Write a MATLAB program to draw the polar plot and to calculate gain margin and phase margin for the open loop system governed by the following transfer function.

G(s)=1/(s(1+s)²)

```
%Program to draw polar plot and compute gain & phase margins

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=1/(s*(1+s)*(1+s))

[num_cof den_cof]=tfdata(Gs);              %determine numerator and
                                            %denominator coeff. of G(s)
w=0.4 : 0.01 : 4;                           %specify frequency range
[re,im,w]=nyquist(num_cof,den_cof,w);       %determine the real and
                                            %imaginary parts of G(jw)

[GM PM]=margin(num_cof, den_cof);           %compute gain & phase margins
disp('Gain margin,');GM
disp('Phase margin in deg,');PM
```

```
z=re+i*im;                                   %convert rectangular
                                            %coordinates to polar
r=abs(z);
theta=angle(z);
polar(theta,r,'k')                           %draw polar plot
```

OUTPUT

```
The given transfer function is,

Transfer function:
                         1
                  ---------------
                  s^3 + 2 s^2 + s

Gain margin,
          GM =
                 2

Phase margin in deg,
                     PM =
                          21.3877

The Polar plot of program 3.5 is shown in fig p3.5.
```

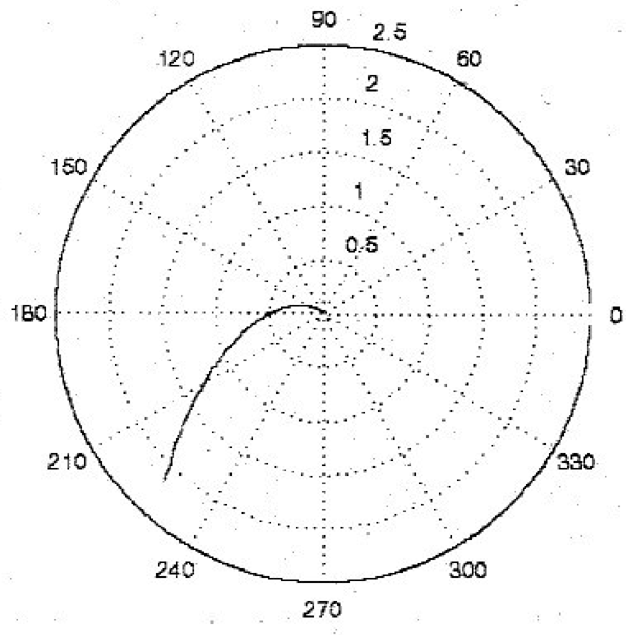


Fig P3.5 : *Polar plot of the open loop system given in problem 3.5.*

PROGRAM 3.6

Write a MATLAB program to draw the polar plot for the open loop system governed by the following transfer function. a) In the frequency range 0.6 to 8 rad/sec b)In the frequency range 5 to 18 rad/sec.

G(s)=1/(S(1+0.2s)(1+0.05s))

```
%program to draw polar plot for two different frequency ranges

clear all
clc
```

```
s=tf('s');
disp('The given transfer function is,');
Gs=1/(s*(1+0.2*s)*(1+0.05*s))

[num_cof den_cof]=tfdata(Gs);   %determine numerator & denominator
                                 %coefficients of G(s)

w1=0.6 :0.001: 8;                               %specify frequency range1
[re1,im1,w1]=nyquist(num_cof,den_cof,w1);%determine the real and
                                                %imaginary part of G(jw)
z1=re1+i*im1;                                   %convert rectangular
r1=abs(z1);                                    %coordinates to polar
theta1=angle(z1);

subplot(2,1,1);polar(theta1,r1,'k');           %draw polar plot for
                                               %frequency range1

w2= 5 :0.001: 18;                               %specify frequency range2
[re2,im2,w2]=nyquist(num_cof,den_cof,w2);%determine the real and
                                                %imaginary part of G(jw)
z2=re2+i*im2;                                   %convert rectangular
r2=abs(z2);                                    %coordinates to polar
theta2=angle(z2);

subplot(2,1,2);polar(theta2,r2,'k');            %draw polar plot for
                                                %frequency range2
```

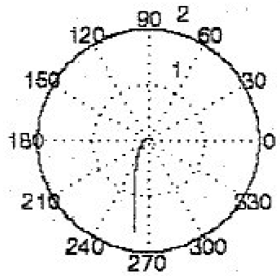


Fig P3.6a : *Polar plot in the frequency range 0.6 to 8 rad/sec.*

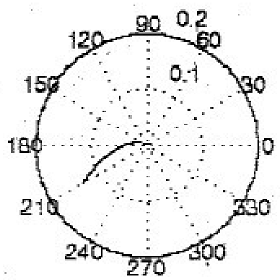


Fig P3.6b : *Polar plot in the frequency range 5 to 18 rad/sec.*

Fig P3.6 : *Polar plots of the open loop system given in problem 3.6.*

OUTPUT

```
The given transfer function is,

Transfer function:
                               1
                    -----------------------
                    0.01 s^3 + 0.25 s^2 + s

The Polar plot of program 3.6 is shown in fig p3.6.
```

PROGRAM 3.7

Write a MATLAB program to draw the polar plot using rectangular to polar coordinates for various values of K for the open loop system governed by the following transfer function.

G(s)=K/(S(1+s)(1+2S))

```
%polar plot for various values of gain,K

clc
s=tf('s');
K=1;
disp('When K=1,the given transfer function is,');
Gs=K/(s*(1+s)*(1+2*s))
[num_cof den_cof]=tfdata(Gs);

for i=1:3;
    if i==1;K=1;[re1,im1]=nyquist([num_cof den_cof],K); end;
    if i==2;K=2;[re2,im2]=nyquist([num_cof den_cof],K);end;
    if i==3;K=25;[re3,im3]=nyquist([num_cof den_cof],K);end;
end

plot(re1,im1,'-k',re2,im2,'-.k',re3,im3,'-k')
axis([-5 1 -5 1]); grid
xlabel('Real axis'); ylabel('Imaginary axis');
text(-3.8,-1.3,'K=1')
text(-3.8,-2.7,'K=2')
text(-1.8,-2.7,'K=25')
```

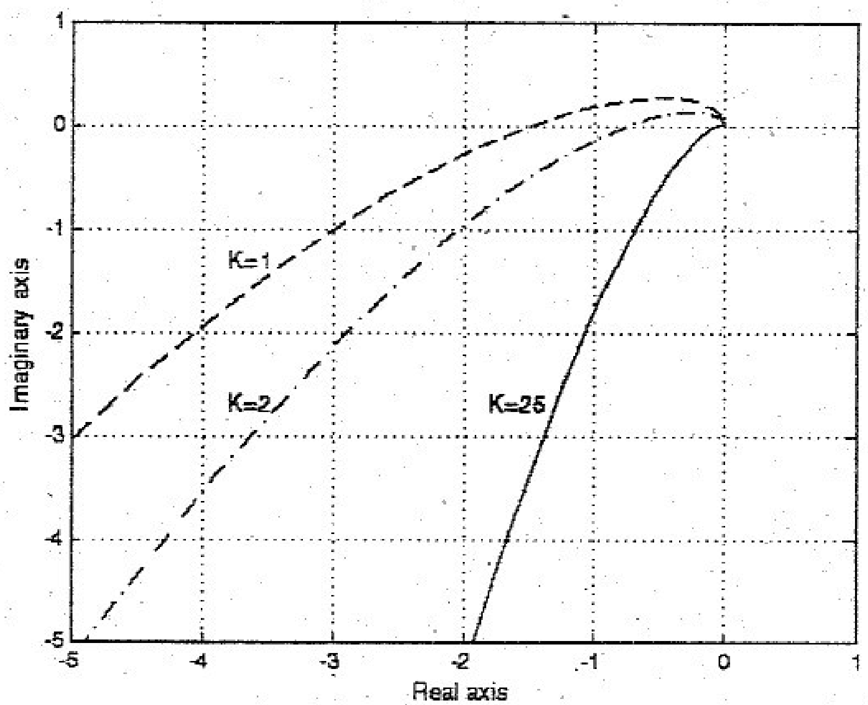


Fig P3.7 : Polar plot of the open loop system for various values of K.

OUTPUT

When K=1, the given transfer function is,

```
Transfer function:
           1
------------------
2 s^3 + 3 s^2 + s
```

PROGRAM 3.8

Write a MATLAB program to draw the polar plot and calculate gain margin and phase margin for the open loop system governed by the following transfer function.

G(s)=1/(s(1+s)²)

```
%program to draw nichols plot on nichols chart

clear all
clc

s=tf('s');
disp('The given transfer function is,');
Gs=20/(s^3+(7*s^2)+10*s)

[num_cof den_cof]=tfdata(Gs);  %determine numerator & denominator
                               %coefficients of G(s)
nichols(Gs);
axis([-180 0 -15 20]);         %specify the range of horizontal and
                               %vertical axis
ngrid
```

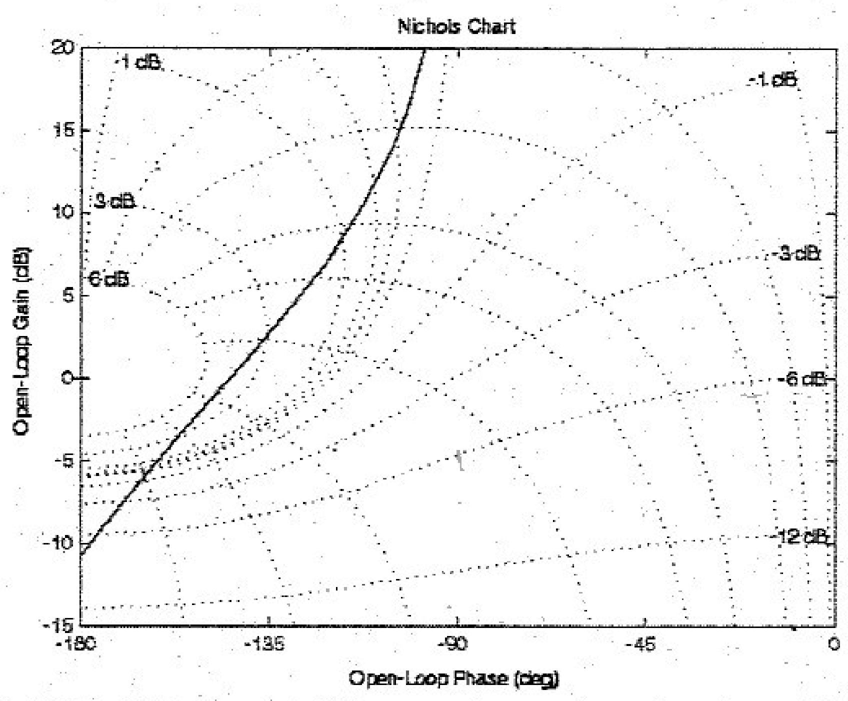


Fig P3.8 : Nichols plot of the open loop system given in problem 3.8.

OUTPUT

```
The given transfer function is,

Transfer function:
              20
      -----------------
      s^3 + 7 s^2 + 10 s
```

3.13 SHORT QUESTIONS AND ANSWERS

Q3.1 ***What is frequency response ?***

The magnitude and phase function of sinusoidal transfer function of a system are real function of frequency ω, and so they are called frequency response.

Q3.2 ***What are advantages of frequency response analysis ?***

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of the open loop frequency response.
2. The practical testing of system can be easily carried with available sinusoidal signal generators and precise measurement equipments.
3. The transfer function of complicated functions can be determined experimentally by frequency response tests.
4. The design and parameter adjustment can be carried more easily.
5. The corrective measure for noise disturbance and parameter variation can be easily carried.
6. It can be extended to certain non-linear systems.

Q3.3 ***What are frequency domain specifications?***

The frequency domain specifications indicates the performance of the system in frequency domain, and they are,

1. Resonant peak, M_r
2. Resonant frequency, ω_r
3. Bandwidth, ω_b
4. Cut-off rate
5. Gain margin, K_g
6. Phase margin, γ

Q3.4 ***Define Resonant Peak?***

The maximum value of the magnitude of closed loop transfer function is called Resonant Peak.

Q3.5 ***What is Resonant frequency?***

The frequency at which the resonant peak occurs is called Resonant frequency. The resonant peak is the maximum value of the magnitude of closed loop transfer function.

Q3.6 ***Define Bandwidth?***

The Bandwidth is the range of frequencies for which the system gain is more than -3db.

Q3.7 ***What is cut-off rate?***

The slope of the log-magnitude curve near the cut-off frequency is called cut-off rate.

Q3.8 ***Define gain margin?***

The gain margin, K_g is defined as the value by which gain of the system has to be increased to drive system to be verge of instability. It is given by the reciprocal of the magnitude of open loop transfer function, at phase cross-over frequency, ω_{pc}. When expressed in decibels, it is given by, the negative of db magnitude of $G(j\omega)$ at phase cross-over frequency.

Gain margin, $K_g = \dfrac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$ and $K_g \text{ in db} = 20 \log \dfrac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = -20 \log |G(j\omega)|_{\omega=\omega_{pc}}$

Q3.9 ***Define phase margin?***

The phase margin, γ is that amount of additional phase lag at the gain cross-over frequency, ω_{gc} required to bring the system to the verge of instability. It is given by, $180° + \phi_{gc}$, where ϕ_{gc} is the phase of $G(j\omega)$ at the gain cross over frequency.

Phase margin, $\gamma = 180^\circ + \phi_{gc}$; where, $\phi_{gc} = \angle G(j\omega)\Big|_{\omega=\omega_{gc}}$

Q3.10 ***What is phase and Gain cross-over frequency?***

The **gain cross over frequency** is the frequency at which the magnitude of the open loop transfer function is unity. The **phase cross over frequency** is the frequency at which the phase of the open loop transfer function is 180°.

Q3.11 ***Write the expression for resonant peak and resonant frequency.***

Resonant peak, $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$; Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

Q3.12 ***Write a short note on the correlation between the time and frequency response?***

Correlation exists between time and frequency response of first or second order systems. The frequency domain specifications can be expressed in terms of the time domain parameters ζ and ω_n. For a peak overshoot in time domain there is a corresponding resonant peak in frequency domain.

For higher order systems, there is no explicit correlation between time and frequency response. But if there is a pair of dominant complex conjugate poles, then the system can be approximated to second order system and the correlation between time and frequency response can be estimated.

Q3.13 ***The damping ratio and natural frequency of oscillation of a second order system is 0.5 and 8 rad/sec respectively. Calculate the resonant peak and resonant frequency?***

$$\text{Resonant peak, } M_r = \frac{1}{2\,\zeta\sqrt{1-\zeta^2}} = \frac{1}{2\times(0.5)\sqrt{1-(0.5)^2}} = 1.154$$

$$\text{Resonant frequency, } \omega_r = \omega_n\sqrt{1-2\zeta^2} = 8\times\sqrt{1-2\times0.5^2} = 5.657 \text{ rad / sec}$$

Q3.14 ***What is Bode plot?***

The bode plot is a frequency response plot of the transfer function of a system. It consists of two plots : *Magnitude plot and Phase plot.*

The magnitude plot is a graph between magnitude of a system transfer function in db and frequency, ω. The phase plot is a graph between the phase or argument of a system transfer function in degrees and the frequency, ω. Usually, both the plots are plotted on a common x-axis in which the frequencies are expressed in logarithmic scale.

Q3.15 ***What is approximate bode plot?***

In approximate bode plot, the magnitude plot of first and second order factors are approximated by two straight lines, which are asymptotes to exact plot. One straight line is at 0db, for the frequency range 0 to ω_c and the other straight line is drawn with a slope of $\pm$ 20n db/dec for frequency range ω_c to ∞. Here ω_c is the corner frequency.

Q3.16 ***Define corner frequency?***

The magnitude plot can be approximated by asymptotic straight lines. The frequencies corresponding to the meeting point of asymptotes are called corner frequency. The slope of the magnitude plot changes at every corner frequency.

Q3.17 ***What are the advantages of Bode Plot?***

1. The magnitudes are expressed in db, and so, a simple procedure is available to add magnitude of each term one by one.

2. The approximate bode plot can be quickly sketched, and the corrections can be made at corner frequencies to get the exact plot.
3. The frequency domain specifications can be easily determined.
4. The bode plot can be used to analyse both open loop and closed loop system.

Q3.18 ***What is the value of error in the approximate magnitude plot of a first order factor at the corner frequency?***

The error in the approximate magnitude plot of a first order factor at the corner frequency is ± 3m db, where m is multiplicity factor. Positive error for numerator factor and negative error for denominator factor.

Q3.19 ***What is the value of error in the approximate magnitude plot of a quadratic factor with $\zeta=1$ at the corner frequency?***

The error is ± 6db, for the quadratic factor with $\zeta=1$. Positive error for numerator factor and negative error for denominator factor.

Q3.20 ***Draw the bode plot of, $G(s) = \dfrac{K}{s^n}$.***

Let $s = j\omega$,

$$\therefore G(j\omega) = \frac{K}{(j\omega)^n}$$

The magnitude of $G(j\omega)$ is unity when $\omega = K^{1/n}$.

The magnitude plot is a straight line with slope of $-20n$ db/dec and passing through $\omega = K^{1/n}$. The Phase plot is straight line parallel to x-axis at $-90n^\circ$.

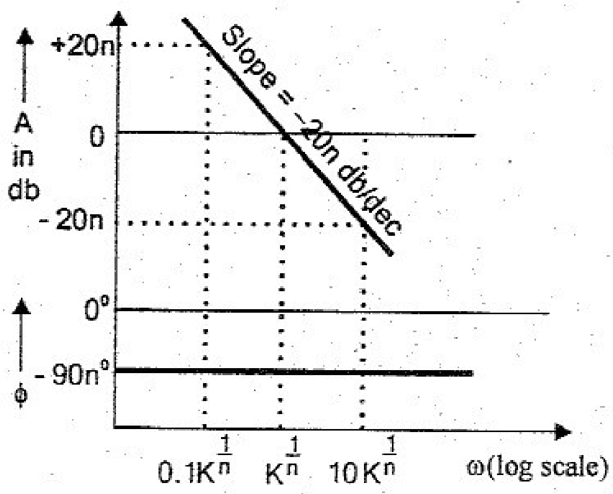


Fig Q3.20 : Bode plot of integral factor, $K/(j\omega)^n$.

Q3.21 ***Sketch the bode plot of $G(s) = 1/(1+sT)$.***

Let $s = j\omega$, $\quad \therefore G(j\omega) = \dfrac{1}{1+j\omega T}$

The corner frequency, $\omega_c = \dfrac{1}{T}$

The magnitude plot is approximated by two straight lines : *one straight line at 0db in the frequency range 0 to ω_c and the other straight line with the slope of -20db/dec in the frequency range ω_c to ∞.* The phase of $G(j\omega)$ varies from 0 to -90° as ω is varied from 0 to ∞. Hence, the phase plot is a curve passing through -45° at the corner frequency.

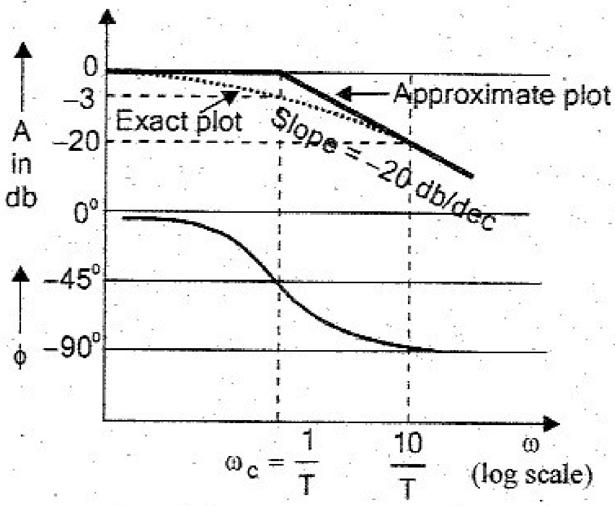


Fig Q3.21 : Bode plot of the factor $\frac{1}{1+j\omega T}$.

Q3.22 What is polar plot?

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle/argument of $G(j\omega)$ on polar or rectangular co-ordinates as ω is varied from zero to infinity.

Q3.23 What is minimum phase system?

The minimum phase systems are systems with minimum phase transfer functions. In minimum phase transfer functions, all poles and zeros will lie on the left half of s-plane.

Q3.24 What is All-Pass systems?

The all pass systems are systems with all pass transfer functions. In all pass transfer functions, the magnitude is unity at all frequencies and the transfer function will have anti-symmetric pole zero pattern (i.e., for every pole in the left half s-plane, there is a zero in the mirror image position with respect to imaginary axis).

Q3.25 What is non-minimum phase transfer function?

A transfer function which has one or more zeros in the right half s-plane is known as non-minimum phase transfer function.

Q3.26 In minimum phase system, how the start and end of polar plot are identified?

For minimum phase transfer functions, with only poles, the type number of the system determines the quadrant in which the polar plot starts, and the order of a system determines the quadrant in which the polar plot ends.

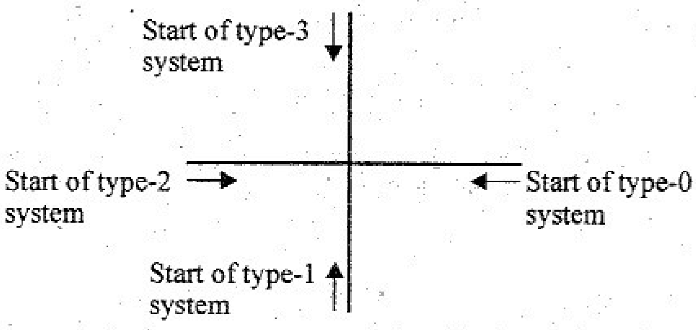


Fig Q3.26a : Start of polar plot of all pole minimum phase system.

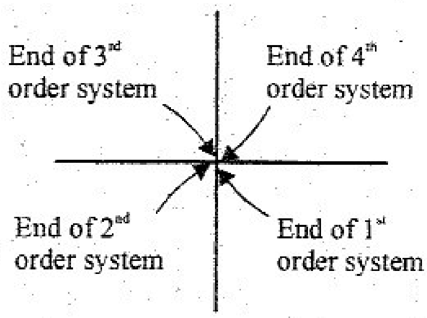


Fig Q3.26b : End of polar plot of all pole minimum phase system.

Q3.27 ***Draw the polar plot of G(s)= 1/(1+sT).***

Let $s = j\omega$, $\therefore G(j\omega) = \dfrac{1}{1+j\omega T} = \dfrac{1}{\sqrt{1+\omega^2T^2}\angle\tan^{-1}\omega T}$

$$= \frac{1}{\sqrt{1+\omega^2T^2}}\angle -\tan^{-1}\omega T$$

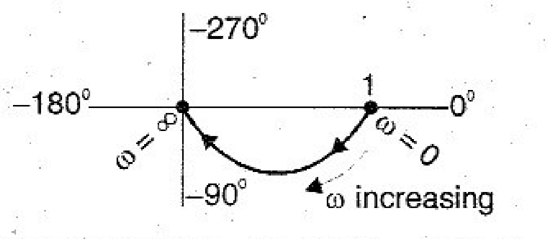


Fig Q3.27 : Polar plot of G(s) = 1/(1+sT).

As $\omega \to 0$, $G(j\omega) \to 1\angle 0°$

As $\omega \to \infty$, $G(j\omega) \to 0\angle -90°$

Q3.28 ***Sketch the polar plot of,*** $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$.

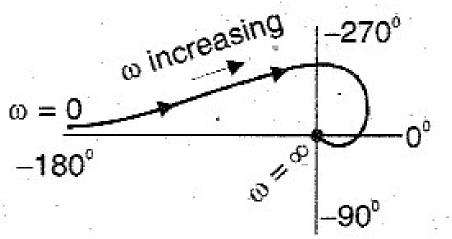


Fig Q3.28 : Polar plot of type-2, 5th order system.

The given system is all pole minimum phase system. The type number of the system is 2 and the order is 5. Hence, the polar plot starts in second quadrant and ends in fourth quadrant.

Q3.29 ***What is Nichols plot?***

The Nichols plot is a frequency response plot of the open loop transfer function of a system. It is a graph between magnitude of $G(j\omega)$ in db and the phase of $G(j\omega)$ in degree, plotted on a ordinary graph sheet.

Q3.30 ***What are M and N circles?***

The magnitude, M of closed loop transfer function with unity feedback will be in the form of circle in complex plane for each constant value of M. The family of these circles are called M-circles.

Let N = tan α, where a is the phase of closed loop transfer function with unity feedback. For each constant value of N, a circle can be drawn in the complex plane. The family of these circles are called N-circles.

Q3.31 ***How closed loop frequency response is determined from open loop frequency response using M and N circles?***

The $G(j\omega)$ locus or polar plot of open loop system is sketched on the standard M and N circles chart. The meeting point of M circle with $G(j\omega)$ locus gives the magnitude of closed loop system. (the frequency being same as that of open loop system). The meeting point of $G(j\omega)$ locus with N-circle gives the value of phase of closed loop system, (frequency being same as that of open loop system).

Q3.32 ***What is Nichols chart?***

The Nichols chart consists of M and N contours superimposed on ordinary graph. Along each M-contour the magnitude of closed loop system, M will be a constant. Along each N-contour, the phase α of closed loop system will be constant. The ordinary graph consists of magnitude in db, marked on the y-axis and the phase in degrees marked on x-axis. The Nichols chart is used to find the closed loop frequency response from the open loop frequency response.

Q3.33 ***How the closed loop frequency response is determined from the open loop frequency response using Nichols chart?***

The $G(j\omega)$ locus or the Nichols plot is sketched on the standard Nichols chart. The meeting point of M-contour with $G(j\omega)$ locus gives the magnitude of closed loop system and the meeting point with N-circle gives the argument/phase of the closed loop system.

Q3.34 ***What are the advantages of Nichols chart?***

1. It is used to find closed loop frequency response from open loop frequency response.
2. The frequency domain specifications can be determined from Nichols chart.
3. The gain of the system can be adjusted to satisfy the given specification.

3.14 EXERCISES

E3.1 *Sketch the bode plot of the following open loop transfer functions and from the plot determine the phase margin and gain margin.*

a) $G(s) = 100(1+0.1s)/s(1+0.2s)\ (1+0.5s)$

b) $G(s) = 50(1+0.1s)/(1+0.01s)\ (1+s)$

c) $G(s) = 30(1+0.1s)/s(1+0.01s)\ (1+s)$

d) $G(s) = s^2(s+10)/(s+5)^2\ (s+0.1)$

e) $G(s) = 40\ (1+s)/(1+5s)\ (s^2+2s+4)$

f) $G(s) = 10(1+s)\ e^{-0.1s}/s(1+0.2s)$

E3.2 *The open loop transfer function of a system is given by $G(s) = K/s(1+0.5s)\ (1+0.2s)$. Using bode plot find the value of K so that (i) The gain margin of the system is 6db and (ii) The phase margin of the system is 25°.*

E3.3 *Sketch the polar plot of the following transfer functions and from the plot, determine the phase margin and gain margin.*

a) $G(s) = 10\ (s+1)/(s+10)^2$

b) $G(s) = 200\ (s+2)/s(s^2+10s+100)$

c) $e^{-0.1s}/s\ (s+1)\ (s+5)$

d) $1/s\ (s+4)\ (s+8)$

E3.4 *The open loop transfer function of a system is given by $G(s) = K/s\ (s^2+s+4)$. Using polar plot, determine the value of K, so that phase margin is 50°. What is the corresponding value of gain margin?*

E3.5 *A unity feedback system has $G(s) = K/s(1+0.1s)$. Using Nichols chart find the value of K so that resonant peak, $M_r=1.4$. Find the corresponding value of ω_r*

E3.6 *The open loop transfer function of unity feedback system is, $G(s) = K/(1+0.05s)\ (1+0.1s)(1+0.3s)$.*

Using Nichols chart find the value of K so that gain margin of the system is 10db. What is the corresponding value of phase margin.

E3.7 *Using Nichols chart determine the closed loop frequency response of the unity feedback system, whose open loop transfer function is, $G(s) = 200\ (s+1)/s(s+10)^2$.*

E3.8 *A unity feedback system has open loop transfer function $G(s) = 54/(1+0.1s)\ (s^2+8s+25)$.*

Using Nichols chart determine the closed loop frequency response. From the closed loop response determine, the resonant peak, resonant frequency and bandwidth.

CHAPTER 4

CONCEPTS OF STABILITY AND ROOT LOCUS

4.1 IMPULSE RESPONSE AND STABILITY

DEFINITIONS OF STABILITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input (or for any changes in input or for any changes in system parameters).

The different definitions of the stability are the following

1. A system is stable, if its output is bounded (finite) for any bounded (finite) input.
2. A system is asymptotically stable, if in the absence of the input, the output tends towards zero (or to the equilibrium state) irrespective of initial conditions.
3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as t approaches infinity.
4. A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory.
5. For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called ***limitedly stable.***
6. If a system output is stable for all variations of its parameters, then the system is called ***absolutely stable system.***
7. If a system output is stable for a limited range of variations of its parameters, then the system is called ***conditionally stable system.***

IMPULSE RESPONSE OF A SYSTEM

Let, M(s) = Closed loop transfer function of a system.

C(s) = Output / Response in s-domain.

R(s) = Input in s-domain

Now, $M(s) = \dfrac{C(s)}{R(s)}$

$\therefore$ Response or Output in s-domain, $C(s) = M(s)\, R(s)$

Now, Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\}$

Input in time domain, $r(t) = \mathcal{L}^{-1}\{R(s)\}$

For an impulse input, $r(t) = \delta(t)$; $\therefore R(s) = \mathcal{L}[\delta(t)] = 1$

$\therefore$ Impulse response $= \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{M(s)\,R(s)\} = \mathcal{L}^{-1}\{M(s)\} = m(t)$ .....(4.1)

Hence, impulse response of a system is the inverse Laplace transform of system transfer function.

The importance of impulse response is that, the output of a system for any arbitrary input can be obtained by convolution of input and impulse response.

i.e., Response, $c(t) = m(t) * r(t)$

where $*$ is the symbol for convolution.

Mathematically the convolution operation is defined as,

$$c(t) = \int_{-\infty}^{+\infty} m(\tau)\; r(t-\tau)\; d\tau \qquad .....(4.2)$$

where t is the dummy variable used for integration.

BOUNDED - INPUT BOUNDED - OUTPUT (BIBO) STABILITY

A linear relaxed system is said to have BIBO stability if every bounded (finite) input results in a bounded (finite) output. A condition for BIBO stability can be obtained from convolution operation defined by equation (4.2).

For a relaxed system the equation (4.2) can be written as,

$$\text{Response,}\quad c(t) = \int_{0}^{\infty} m(\tau)\; r(t-\tau)\; d\tau \qquad .....(4.3)$$

> ***Note :*** *A relaxed system is one in which the initial conditions are zero. Hence the limits of integration is from 0 to ∞.*

If the input r(t) is bounded then there exists a constant A_1, such that $|r(t)| \le A_1 < \infty$. The condition for bounded output for this bounded input condition can be derived as follows.

On taking the absolute value on both sides of equation (4.3), we get,

$$|c(t)| = \left| \int_{0}^{\infty} m(\tau)\; r(t-\tau)\; d\tau \right| \qquad .....(4.4)$$

Since the absolute value of an integral is not greater than the integral of the absolute value of the integrand the equation (4.4) can be written as,

$$|c(t)| \le \int_{0}^{\infty} |m(\tau)\, r(t-\tau)|\, d\tau \;\Rightarrow\; |c(t)| \le \int_{0}^{\infty} |m(\tau)|\, |r(t-\tau)|\, d\tau \;\Rightarrow\; |c(t)| \le \int_{0}^{\infty} |m(\tau)|\, A_1 d\tau$$

$$\therefore\; |c(t)| \;\le\; A_1 \int_{0}^{\infty} |m(\tau)|\; d\tau$$

For bounded input, a constant exists such that, $\|r(t-\tau)\| \le A_1$.

If the output c(t) is bounded then there exists a constant A_2 such that $|c(t)| \le A_2 < \infty$.

$$\therefore\; A_1 \int_{0}^{\infty} |m(\tau)|\; d\tau \le A_2 < \infty \qquad .....(4.5)$$

The above condition is satisfied if, $\int_0^{\infty} |m(\tau)|\, d\tau < \infty$

τ is a dummy variable and so can be replaced by t

Hence for bounded output, $\int_0^{\infty} |m(t)|\, dt < \infty$ .....(4.6)

Therefore we can conclude that a system with impulse response m(t) is BIBO stable if and only if the impulse response is absolutely integrable (i.e., $\int_0^{\infty} |m(t)|\, dt$ is finite. This means that area under the absolute value curve of the impulse response m(t) evaluated from t = 0 to t = ∞ must be finite).

4.2 LOCATION OF POLES ON s-PLANE FOR STABILITY

The closed loop transfer function, M(s) can be expressed as a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called characteristic equation. The roots of characteristic equation are poles of closed loop transfer function.

For BIBO stability the integral of impulse response should be finite, which implies that the impulse response should be finite as t tends to infinity. *[The impulse response is the inverse Laplace transform of the transfer function]*. This requirement for stability can be linked to the location of roots of characteristic equation in the s-plane.

The closed loop transfer function M(s) can be expressed as a ratio of two polynomials in s as shown in equation (4.7).

$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + ... + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1} s + a_n} \quad .....(4.7)$$

$$= \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......(s+z_m)}{(s+p_1)\,(s+p_2)\,(s+p_3)\,.......(s+p_n)} \quad .....(4.8)$$

The roots of numerator polynomial z_1, z_2, ...z_n are zeros. The roots of denominator polynomial p_1, p_2, ...p_n are poles. The denominator polynomial is the characteristic equation and so the poles are roots of characteristic equation.

By partial fraction expansion we can write,

$$M(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + ... + \frac{A_n}{s+p_n} \quad .....(4.9)$$

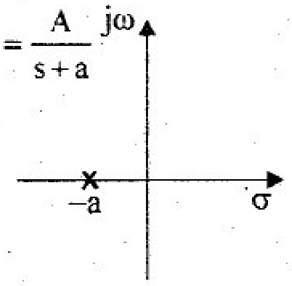
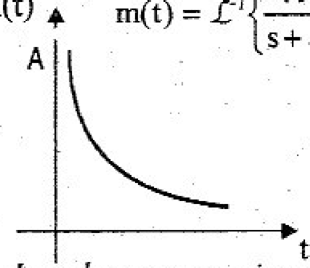
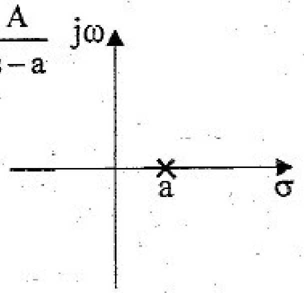
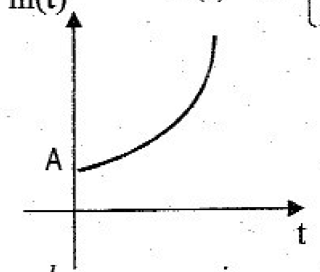
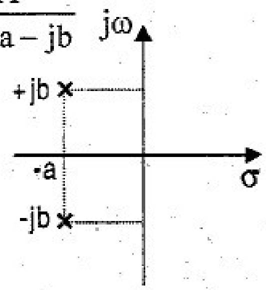
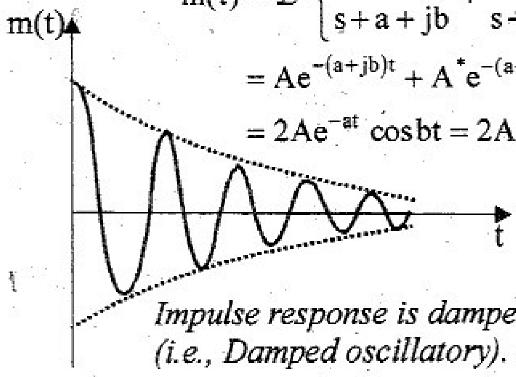
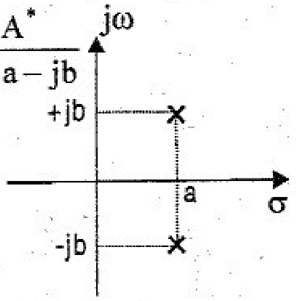
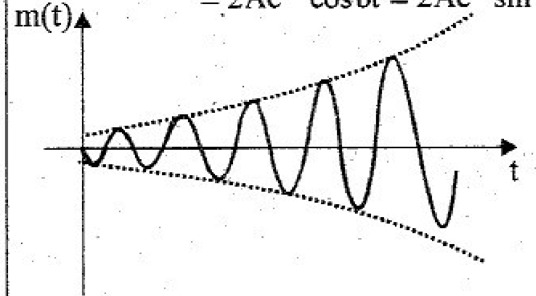
The roots (or poles) p_1, p_2, p_3, ... p_n may be at origin or lying on imaginary axis or lying on right or left half of s-plane. The impulse response is given by inverse Laplace transform of M(s). The inverse Laplace transform of each term of M(s) depends on the location of roots (or poles) in s-plane. The impulse response of various types of M(s) are shown in table-4.1.

From table 4.1, the following conclusions are drawn based on the location of roots of characteristic equation.

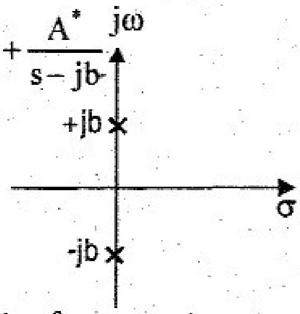
1. If all the roots of characteristic equation have negative real parts (i.e., lying on left half s-plane) then the impulse response is bounded (i.e., it decreases to zero as t tends to ∞).

   Hence $\int_0^{\infty} |m(t)|\, dt$ is finite and the system is bounded-input bounded-output stable.

TABLE-4.1

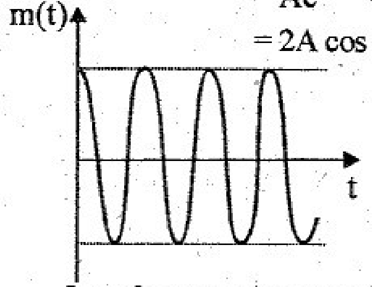
Transfer function, M(s) and location of roots on s-plane	Impulse response, m(t)
$M(s)=\dfrac{A}{s+a}$  *Root on negative real axis*	$m(t)=\mathcal{L}^{-1}\left\{\dfrac{A}{s+a}\right\}=Ae^{-at}$  *Impulse response is exponentially decaying. Stable system.*
$M(s)=\dfrac{A}{s-a}$  *Root on positive real axis*	$M(s)=\mathcal{L}^{-1}\left\{\dfrac{A}{s-a}\right\}=Ae^{+at}$  *Impulse response is exponentially increasing. Unstable system.*
$M(s)=\dfrac{A}{s+a+jb}+\dfrac{A^*}{s+a-jb}$  *Complex conjugate roots on left half of s-plane*	$m(t)=\mathcal{L}^{-1}\left\{\dfrac{A}{s+a+jb}+\dfrac{A^*}{s+a-jb}\right\}$ $=Ae^{-(a+jb)t}+A^*e^{-(a-jb)t}$ $=2Ae^{-at}\cos bt=2Ae^{-at}\sin\left(bt+90^\circ\right)$  *Impulse response is damped sinusoidal (i.e., Damped oscillatory). Stable system*
$M(s)=\dfrac{A}{s-a+jb}+\dfrac{A^*}{s-a-jb}$  *Complex conjugate roots on right half of s-plane*	$m(t)=\mathcal{L}^{-1}\left\{\dfrac{A}{s-a+jb}+\dfrac{A^*}{s-a-jb}\right\}$ $=Ae^{-(-a+jb)t}+A^*e^{-(-a-jb)t}$ $=2Ae^{at}\cos bt=2Ae^{at}\sin\left(bt+90^\circ\right)$  *Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.*

$$M(s) = \frac{A}{s+jb} + \frac{A^*}{s-jb}$$



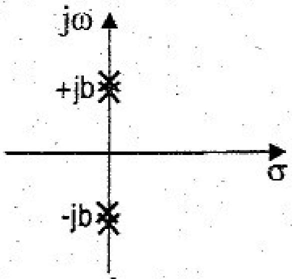
Single pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+jb} + \frac{A^*}{s-jb}\right\}$$
$$= Ae^{-jbt} + A^*e^{+jbt}$$
$$= 2A\cos bt = 2A\sin(bt + 90°)$$



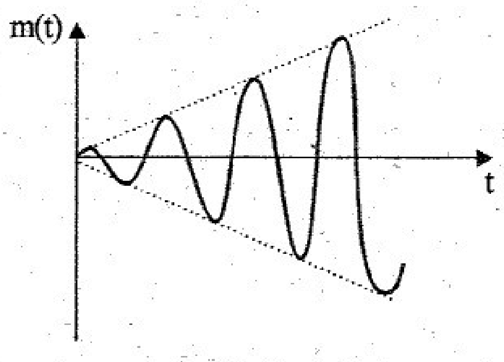
Impulse response is oscillatory
Marginally stable

$$M(s) = \frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}$$



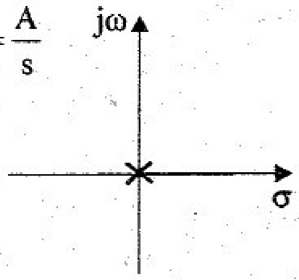
Double pair of roots on imaginary axis

$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2} + \frac{A^*}{(s-jb)^2}\right\}$$
$$= At\,e^{-jbt} + A^*\,t\,e^{+jbt}$$
$$= 2At\cos bt = 2At\sin(bt + 90°)$$



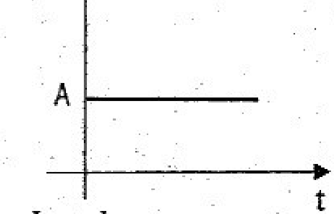
Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.

$$M(s) = \frac{A}{s}$$

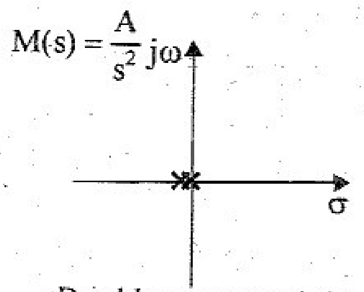


Single root at origin

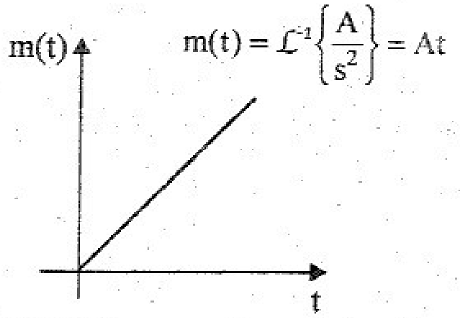
$$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = A$$



Impulse response is constant.
Marginally stable system.



Double root at origin



Impulse reponse linearly increases with time. Unstable system

2. If any root of the characteristic equation has a positive real part (i.e., lying on right half s-plane) then impulse response is unbounded, (i.e., it increases to ∞ as t tends to ∞). Hence $\int_0^{\infty} |m(t)|\, dt$ is infinite and so system is unstable.
3. If the characteristic equation has repeated roots on the imaginary axis then impulse response is unbounded (i.e., it increases to ∞ as t tends to ∞).

   Hence $\int_0^{\infty} |m(t)|\, dt$ is infinite and so the system is unstable.
4. If one or more non - repeated roots of the characteristic equation are lying on the imaginary axis, then impulse response is bounded (i.e., it has constant amplitude oscillations) but is infinite and so the system is unstable.
5. If the characteristic equation has single root at origin then the impulse response is bounded (i.e, it has constant amplitude) but $\int_0^{\infty} |m(t)|\, dt$ is infinite and so the system is unstable.
6. If the characteristic equation has repeated roots at origin then the impulse response is unbounded (i.e., it linearly increases to infinity as t tends to ∞) and so the system is unstable.
7. In system with one or more non-repeated roots on imaginary axis or with single root at origin, the output is bounded for bounded inputs except for the inputs having poles matching the system poles. These cases may be treated as acceptable or non-acceptable. Hence when the system has non repeated poles on imaginary axis or single pole at origin, it is referred as limitedly or marginally stable system.

In summary, the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

1. ***If all the roots of characteristic equation has negative real parts, then the system is stable.***
2. ***If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.***
3. ***If the condition (i) is satisfied except for the presence of one or more non repeated roots on the imaginary axis, then the system is limitedly or marginally stable.***

In order to ascertain the stability of a system, it is necessary to determine if any of the roots of the characteristic equation lie in the right half s-plane. The characteristic equation is given by the denominator polynomial of closed loop transfer function, [equation (4.7)].

Consider the n^{th} order characteristic equation shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_{n-1} s + a_n = 0 \quad .....(4.10)$$

Let the roots of n^{th} order characteristic equation [equation (4.10)] be $s = r_1, r_2, ...r_n$. These roots are functions of the coefficients $a_0, a_1, a_2, ... a_{n-1}, a_n$.

Consider a second order polynomial,

$$a_0 s^2 + a_1 s + a_2 = a_0 \left(s^2 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} \right)$$

$$= a_0 (s - r_1)(s - r_2)$$

$$= a_0 s^2 - a_0 (r_1 + r_2) s + a_0 r_1 r_2 \quad .....(4.11)$$

Consider a third order polynomial

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = a_0 \left(s^3 + \frac{a_1}{a_0} s^2 + \frac{a_2}{a_0} s + \frac{a_3}{a_0} \right)$$

$$= a_0 (s - r_1)(s - r_2)(s - r_3)$$

$$= a_0 s^3 - a_0 (r_1 + r_2 + r_3) s^2$$

$$+ a_0 (r_1 r_2 + r_1 r_3 + r_2 r_3) s - a_0 r_1 r_2 r_3 \quad .....(4.12)$$

On extending this expansion to the n^{th} order polynomial, we get.

$$a_0 s^n + a_1 s^{n-1} + ...+ a_{n-1} s + a_n = a_0 s^n - a_0 (\text{sum of all the roots})\, s^{n-1}$$

$$+ a_0 \left(\begin{array}{l} \text{sum of the products of the roots} \\ \text{taken 2 at a time} \end{array} \right) s^{n-2}$$

$$- a_0 \left(\begin{array}{l} \text{sum of the products of the roots} \\ \text{taken 3 at a time} \end{array} \right) s^{n-3}$$

$$+ .... + a_0 (-1)^n \,(\text{Product of all the n roots}) \quad .....(4.13)$$

If all the roots of a polynomial are real and in the left half of s-plane, then all r_i in equations (4.11) and (4.12) are real and negative. Therefore all polynomial coefficients are positive. This characteristic also applies to the general case of equation (4.13). If atleast one root is in the right half of s-plane then some of the coefficients will be negative. Also, it can be observed that if all the roots are in the left half of s-plane, no coefficient can be zero.

Since the characteristic polynomial coefficients are real, the complex roots should occur as conjugate pairs. From equation (4.13) it can be inferred that when polynomial coefficients are formed, the imaginary parts of roots/products of roots will cancel. Therefore, if all roots occur in the left half plane, (whether it is complex or real) then all coefficients of the general polynomial of equation (4.13) will be positive. Presence of a negative coefficient implies that there is atleast one root in the right half of s- plane.

A zero coefficient indicates presence of complex-conjugate roots on the imaginary axis and/or one or more roots in the right half of s- plane.

In summary, following conclusions can be made about coefficients of characteristic polynomial.

1. ***If all the coefficients are positive and if no coefficient is zero, then all the roots are in the left half of s- plane.***
2. ***If any coefficient a_i is equal to zero then, some of the roots may be on the imaginary axis or on the right half of s- plane.***

3. ***If any coefficient a_i is negative then atleast one root is in the right half of s- plane.***

It can be concluded that the absence or negativeness of any of the coefficients of a characteristic polynomial indicates that the system is either unstable or at most marginally stable. Thus ***the necessary condition for stability of the system is that all the coefficients of its characteristic polynomial be positive.*** If any coefficient is zero/negative, we can immediately say that the system is unstable.

In order for all the roots to have negative real parts, it is necessary that all of the coefficients of characteristic equation be positive, but it is not sufficient, because there may be roots in the right half plane and/or on the imaginary axis, even when coefficients are positive.(i.e., when roots have negative real part, then all the coefficients of characteristic polynomial will be positive, but the reverse condition is not true always).

$$\text{Roots with negative real part.} \underset{\text{not always true}}{\overset{\text{always true}}{\rightleftarrows}} \text{All coefficients positive.}$$

Hence, when all the coefficients are positive, the system may or may not be stable, because there may be roots in the right half plane and/or on the imaginary axis.

For example, consider the characteristic polynomial with all positive coefficients,

$$s^3 + s^2 + 2s + 8 = 0.$$

The characteristic polynomial can be written as,

$$(s^3+s^2+2s+8) = (s+2)\left(s-\frac{1}{2}-j\frac{\sqrt{15}}{2}\right)\left(s-\frac{1}{2}+j\frac{\sqrt{15}}{2}\right)=0$$

Now the roots are,

$$s=-2, \quad +\frac{1}{2}+j\frac{\sqrt{15}}{2}, \quad +\frac{1}{2}-j\frac{\sqrt{15}}{2}$$

The coefficients of the polynomial are all positive, but two roots have positive real part and so will lie on on right half of s-plane, therefore the system is unstable.

4.3 ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Eventhough the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.

A. Hurwitz and E.J. Routh independently published the method of investigating the sufficient conditions of stability of a system. The Hurwitz criterion is in terms of determinants and Routh criterion is in terms of array formulation. The Routh stability criterion is presented here.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation, into a schedule, called the Routh array as shown below.

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n = 0, \text{ where } a_0 > 0,$$

$$\begin{array}{llllllll}
s^n & : & a_0 & a_2 & a_4 & a_6 & a_8 & \dots\dots \\
s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & a_9 & \dots\dots \\
s^{n-2} & : & b_0 & b_1 & b_2 & b_3 & b_4 & \dots\dots \\
s^{n-3} & : & c_0 & c_1 & c_2 & c_3 & c_4 & \dots\dots \\
 & & & & & & & \\
s^1 & : & g_0 & & & & & \\
s_0 & : & h_0 & & & & &
\end{array}$$

The Routh stability criterion can be stated as follows.

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

Note : If the order of sign of first column element is +, +, –, + and +. Then + to – is considered as one sign change and – to + as another sign change.

CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be,

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + a_3s^{n-3} + \dots a_{n-1}s^1 + a_ns^0$$

The coefficients of the polynomial are arranged in two rows as shown below.

$$\begin{array}{lllllll}
s^n & : & a_0 & a_2 & a_4 & a_6 & \dots\dots \\
s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & \dots\dots
\end{array}$$

When n is even, the s^n row is formed by coefficients of even order terms (i.e., coefficients of even powers of s) and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

When n is odd, the s^n row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s) and s^{n-1} row is formed by coefficients of even order terms(i.e., coefficients of even powers of s) .

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of Routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of Routh array as shown below.

$$s^{n-x} \; : x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \ldots\ldots$$

$$s^{n-x-1} : y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \ldots\ldots$$

Let the next row be,

$$s^{n-x-2} : z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \ldots\ldots$$

The elements of s^{n-x-2} row are given by,

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0 x_1 - y_1 x_0}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0 x_2 - y_2 x_0}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0 x_3 - y_3 x_0}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0 x_4 - y_4 x_0}{y_0}$$

$$z_4 = \frac{(-1)\begin{vmatrix} x_0 & x_5 \\ y_0 & y_5 \end{vmatrix}}{y_0} = \frac{y_0 x_5 - y_5 x_0}{y_0} \quad \text{and so on.}$$

The elements z_0, z_1, z_2, z_3,.... are computed for all possible computations as shown above.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a positive constant to simplify the computational work.

In the construction of Routh array one may come across the following three cases.

Case-I : Normal Routh array (Non-zero elements in the first column of routh array).

Case-II : A row of all zeros.

Case-III : First element of a row is zero but some or other elements are not zero.

Case-I : Normal routh array

In this case, there is no difficulty in forming routh array. The routh array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the system can be estimated.

In this case,

1. If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

2. If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II : A row of all zeros

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is also called ***auxiliary polynomial.*** The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The fig 4.1 shows the roots of an even polynomial.

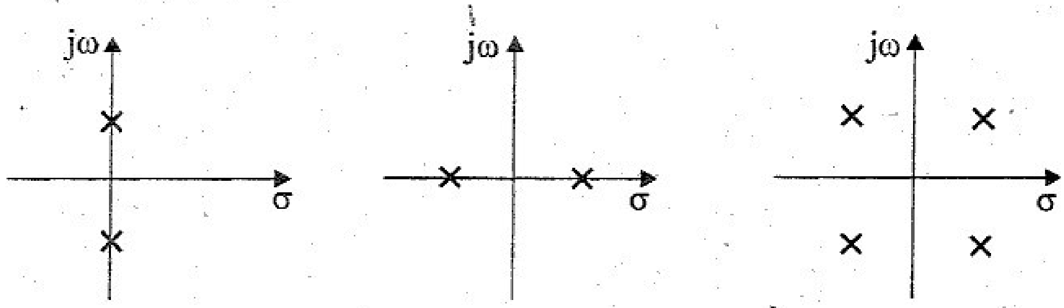


Fig 4.1 : *The roots of an even polynomial.*

The case-II polynomial can be analyzed by any one of the following two methods.

METHOD-1

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner (as that of case-I) and the array is interpreted as follows.
   a. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.
   b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

METHOD-2

1. Determine the auxiliary polynomial, A(s).
2. Divide the characteristic equation by auxiliary polynomial.

3. Construct Routh array using the coefficients of quotient polynomial.
4. The array is interpretted as follows.
   a. If there are sign changes in the first column of routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s-plane is given by number of sign changes in first column of routh array.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex.

      The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots of auxiliary polynomial with positive real part. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

   b. If there is no sign change in the first column of routh array of quotient polynomial then the system is limitedly or marginally stable. Since there is no sign change all the roots of quotient polynomial are lying on the left half of s-plane.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

Case-III : First element of a row is zero

While constructing routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \rightarrow \in$ and complete the construction of array in the usual way (as that of case-I)

Finally let $\in \rightarrow 0$ and determine the values of the elements of the array which are functions of $\in$. The resultant array is interpreted as follows.

> ***Note :*** *If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case-II polynomial. Even if there is a single element zero on s^1 row, it is considered as a row of all zeros.*

a. If there is no sign change in first column of routh array and if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

b. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

c. If there is a row of all zeros after letting $\in \rightarrow 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial.

EXAMPLE 4.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	18	5	.... Row-1
s^3	:	8	16		.... Row-2

The elements of s^3 row can be divided by 8 to simplify the computations.

s^4	:	1	18	5	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	16	5		.... Row-3
s^1	:	1.7			.... Row-4
s^0	:	5			.... Row-5

Column-1

$$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$$

$$s^2 : \ 16 \quad 5$$

$$s^1 : \frac{16\times2-5\times1}{16}$$

$$s^1 : \ 1.6875 \approx 1.7$$

$$s^0 : \frac{1.7\times5-0\times16}{17}$$

$$s^0 : \ 5$$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EXAMPLE 4.2

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is, $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^6	:	1	8	20	16	.... Row-1
s^5	:	2	12	16		.... Row-2

The elements of s^5 row can be divided by 2 to simplify the calculations.

s^6	:	1	8	20	16	.... Row-1
s^5	:	1	6	8		....Row-2
s^4	:	1	6	8		....Row-4
s^3	:	0	0			....Row-4
s^3	:	1	3			....Row-4
s^2	:	3	8			....Row-5
s^1	:	0.33				....Row-6
s^0	:	8				....Row-7

Column-1

$s^4 : \frac{1\times 8-6\times 1}{1} \quad \frac{1\times 20-8\times 1}{1} \quad \frac{1\times 16-0\times 1}{1}$

$s^4 :\quad 2 \quad 12 \quad 16$

divide by 2

$s^4 :\quad 1 \quad 6 \quad 8$

$s^3 : \frac{1\times 6-6\times 1}{1} \quad \frac{1\times 8-8\times 1}{1}$

$s^3 :\quad 0 \quad 0$

The auxiliary equation is, $A = s^4+6s^2+8$. On differentiating A with respect to s we get,

$$\frac{dA}{ds} = 4s^3 + 12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row.

$s^3 :\quad 4 \quad 12$

divide by 4

$s^3 :\quad 1 \quad 3$

$s^2 : \frac{1\times 6-3\times 1}{1} \quad \frac{1\times 8-0\times 1}{1}$

$s^2 :\quad 3 \quad 8$

$s^1 : \frac{3\times 3-8\times 1}{3}$

$s^1 :\ 0.33$

$s^0 : \frac{0.33\times 8-0\times 3}{0.33}$

$s^0 :\ 8$

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4+6s^2+8=0$$

Let, $s^2 = x$

$$\therefore x^2+6x+8=0$$

The roots of quadratic are, $x = \frac{-6\pm\sqrt{6^2-4\times 8}}{2}$

$$= -3\pm 1 = -2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= +j\sqrt{2}, -j\sqrt{2}, +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

EXAMPLE 4.3

Construct Routh array and determine the stability of the system represented by the characteristic equation, $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^5+s^4+2s^3+2s^2+3s+5=0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5	:	1	2	3	.... Row-1
s^4	:	1	2	5	.... Row-2

$$\begin{array}{llccl}
s^3 & : & \in & -2 & \text{.... Row-3} \\
s^2 & : & \dfrac{2\in+2}{\in} & 5 & \text{.... Row-4} \\
s^1 & : & \dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2} & & \text{.... Row-5} \\
s^0 & : & 5 & & \text{.... Row-6}
\end{array}$$

$$s^3: \frac{1\times2-2\times1}{1} \quad \frac{1\times3-5\times1}{1}$$

$$s^3: \quad 0 \qquad -2$$

Replace 0 by $\in$

$$s^3: \quad \in \qquad -2$$

$$s^2: \frac{\in\times2-(-2\times1)}{\in} \quad \frac{\in\times5-0\times1}{\in}$$

$$s^2: \frac{2\in+2}{\in} \qquad 5.$$

$$s^1: \frac{\dfrac{2\in+2}{\in}\times(-2)-(5\times\in)}{\dfrac{2\in+2}{\in}}$$

$$s^1: \frac{-(5\in^2+4\in+4)}{2\in+2}$$

$$s^0: \frac{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}\times5-0\times\dfrac{2\in+2}{\in}}{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}}$$

$$s^0: 5$$

On letting $\in\to0$, we get

			Column-1			
	s^5	:	1	2	3	.... Row-1
	s^4	:	1	2	5	.... Row-2
	s^3	:	0	-2		.... Row-3
+ to −	s^2	:	∞	5		.... Row-4
− to +	s^1	:	-2			.... Row-5
	s^0	:	5			.... Row-6

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

RESULT

(a). The system is unstable.

(b). Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

EXAMPLE 4.4

By routh stability criterion determine the stability of the system represented by the characteristic equation, $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is, $9s^5-20s^4+10s^3-s^2-9s-10=0$

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right half of s-plane. Hence the system is unstable. The routh array can be constructed to find the number of roots lying on right half of s-plane.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

			Column-1			
+ to −	s^5	:	9	10	−9	.... Row-1
− to +	s^4	:	-20	−1	−10	.... Row-2
+ to −	s^3	:	9.55	−13.5		.... Row-3
	s^2	:	−29.3	−10		.... Row-4
	s^1	:	−16.8			.... Row-5
	s^0	:	−10			.... Row-6

$$s^3: \frac{-20\times10-(-1)\times9}{-20} \quad \frac{-20\times(-9)-(-10)\times9}{-20}$$

$$s^3: 9.55 \quad -13.5$$

$$s^2: \frac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \frac{9.55\times(-10)}{9.55}$$

$$s^2: -29.3 \quad -10$$

By examining the elements of I[st] column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

$$s^1 : \frac{-29.3 \times (-13.5) - (-10) \times 9.55}{-29.3}$$

$$s^1 : -16.8$$

$$s^0 : \frac{-16.8 \times (-10)}{-16.8}$$

$$s^0 : -10$$

RESULT

(a). The system is unstable.

(b). Three roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.5

The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

METHOD-I

The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7[th] order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

	s^7	:	1	24	24	23	.... Row-1
	s^6	:	3	8	8	5	.... Row-2
	s^5	:	1	1	1		.... Row-3
	s^4	:	1	1	1		.... Row-4
	s^3	:	0	0			.... Row-5
	s^3	:	2	1			.... Row-5
	s^2	:	0.5	1			.... Row-6
+ to −	s^1	:	-3				.... Row-7
− to +	s^0	:	1				.... Row-8
			Column-1				

$$s^5 : \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 24 - 8\times 1}{3} \quad \frac{3\times 23 - 5\times 1}{3}$$

s^5 : 21.33   21.33   21.33

Divide by 21.33

s^5 : 1   1   1

$$s^4 : \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 8 - 1\times 3}{1} \quad \frac{1\times 5 - 0\times 3}{1}$$

s^4 : 5   5   5

Divide by 5

s^4 : 1   1   1

$$s^3 : \frac{1\times 1 - 1\times 1}{1} \quad \frac{1\times 1 - 1\times 1}{1}$$

s^3 : 0   0

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicates the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$

Put, $s^2 = x$ in the auxiliary equation,

$$s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The auxiliary polynomial is,

$$A = s^4 + s^2 + 1$$

Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

s^3 : 4   2

Divide by 2

s^3 : 2   1

The roots of quadratic are, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$= 1\angle 120°$ or $1\angle -120°$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120°}$ or $\pm\sqrt{1\angle -120°}$

$= \pm\sqrt{1}\angle 120°/2$ or $\pm\sqrt{1}\angle -120°/2$

$= \pm 1\angle 60°$ or $\pm 1\angle -60°$

$= \pm(0.5 + j0.866)$ or $\pm(0.5 - j0.866)$

$s^2 : \frac{2\times1-1\times1}{2} \quad \frac{2\times1-0\times1}{2}$

$s^2 : \quad 0.5 \qquad 1$

$s^1 : \frac{0.5\times1-1\times2}{0.5}$

$s^1 : -3$

$s^0 : \frac{-3\times1}{-3}$

$s^0 : 1$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

METHOD-II

The characteristic equation is, $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5

$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$

$s^5 : 21.33 \qquad 21.33 \qquad 21.33$

Divide by 21.33

$s^5 : \quad 1 \qquad 1 \qquad 1$

$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$

$s^4 : \quad 5 \qquad 5 \qquad 5$

Divide by 5

$s^4 : \quad 1 \qquad 1 \qquad 1$

$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$

$s^3 : \quad 0 \qquad 0$

Since we get a row of zeros, there exists an even polynomial, the even polynomial is nothing but, the auxiliary polynomial. The auxiliary polynomial is,

$s^4 + s^2 + 1 = 0$

Divide the characteristic equation by auxiliary polynomial to get the quotient polynomial.

The characteristic polynomial can be expressed as a product of quotient polynomial and auxiliary polynomial.

$$\therefore s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$$

$$\Downarrow$$

$$\underbrace{(s^4 + s^2 + 1)}_{\text{Even polynomial}}\ \underbrace{(s^3 + 9s^2 + 23s + 15)}_{\text{Quotient polynomial}} = 0$$

The routh array is constructed for quotient polynomial as shown below.

$$\begin{array}{lcll} s^3 & : & 1 & 23 \\ s^2 & : & 9 & 15 \end{array}$$

Divide s^2 row by 3,

$$\begin{array}{lcll} s^3 & : & 1 & 23 \\ s^2 & : & 3 & 5 \\ s^1 & : & 21.33 & \\ s^0 & : & 5 & \end{array}$$

Column-1

$$s^1 : \frac{3 \times 23 - 5 \times 1}{3}$$

$$s^1 : 21.33$$

$$s^0 : \frac{21.33 \times 5 - 0 \times 3}{21.33}$$

$$s^0 : 5$$

$$\begin{array}{r|l} & s^3 + 9s^2 + 23s + 15 \quad \text{(Quotient Polynomial)} \\ \hline s^4 + s^2 + 1 \ \text{(Even Polynomial)} & s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 \\ & \underset{(-)}{s^7} \quad \underset{(-)}{+ s^5} \quad \underset{(-)}{+ s^3} \\ \hline & 9s^6 + 23s^5 + 24s^4 + 23s^3 + 24s^2 + 23s + 15 \\ & \underset{(-)}{9s^6} + \underset{(-)}{9s^4} \underset{(-)}{+ 9s^2} \\ \hline & 23s^5 + 15s^4 + 23s^3 + 15s^2 + 23s + 15 \\ & \underset{(-)}{23s^5} \underset{(-)}{+ 23s^3} \underset{(-)}{+ 23s} \\ \hline & 15s^4 + 15s^2 + 15 \\ & \underset{(-)}{15s^4} \underset{(-)}{+ 15s^2} \underset{(-)}{+ 15} \\ \hline & 0 \end{array}$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$.

Put, $s^2 = x$ in the auxiliary equation. $\quad s^4 + s^2 + 1 = x^2 + x + 1 = 0$

The roots of quadratic are, $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2} = 1\angle 120^\circ$ or $1\angle -120^\circ$

$$\begin{array}{lll} \text{But } s^2 = x, \quad \therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ} & \text{or} & \pm\sqrt{1\angle -120^\circ} \\ = \pm\sqrt{1}\angle 120^\circ/2 & \text{or} & \pm\sqrt{1}\angle -120^\circ/2 \\ = \pm 1\angle 60^\circ & \text{or} & \pm 1\angle -60^\circ \\ = \pm(0.5 + j0.866) & \text{or} & \pm(0.5 - j0.866) \end{array}$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by the roots of auxiliary polynomial and the roots of quotient polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so system is unstable. The remaining five roots are lying on left half of s-plane.

EXAMPLE 4.6

The characteristic polynomial of a system is $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$. Determine the location of roots on the s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is, $s^7 + 5s^6 + 9s^5 + 9s^4 + 4s^3 + 20s^2 + 36s + 36 = 0$.

The given characteristic polynomial is 7^{th} order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row of array using the coefficients of even powers of s as shown below.

s^7 : 1 9 4 36 .... Row-1

s^6 : 5 9 20 36 .... Row-2

Divide s^6 row by 5 to simplify the computations.

s^7 : 1 9 4 36 .... Row-1

s^6 : 1 1.8 4 7.2 .... Row-2

s^5 : 1 0 4 .... Row-3

s^4 : 1 0 4 .... Row-4

s^3 : 0 0 .... Row-5

$$s^5 : \frac{1\times 9-1.8\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 36-7.2\times 1}{1}$$

s^5 : 7.2 0 28.8

Divide by 7.2

s^5 : 1 0 4

$$s^4 : \frac{1\times 1.8-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 7.2-0\times 1}{1}$$

s^4 : 1.8 0 7.2

Divide by 1.8

s^4 : 1 0 4

$$s^3 : \frac{1\times 0-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1}$$

s^3 : 0 0

The row of all zeros indicate the existence of even polynomial, which is also the auxiliary polynomial. The auxiliary polynomial is, $s^4+4=0$. Divide the characteristic equation by auxiliary equation to get the quotient polynomial.

$$
\begin{array}{l}
\qquad\qquad s^3+5s^2+9s+9 \\
s^4+4 \;\big)\; s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36 \\
\qquad\quad\;\; s^7 \qquad\qquad\qquad\;\; +4s^3 \\
\hline
\qquad\qquad 5s^6+9s^5+9s^4+20s^2+36s+36 \\
\qquad\qquad 5s^6 \qquad\qquad\quad +20s^2 \\
\hline
\qquad\qquad\qquad 9s^5+9s^4+36s+36 \\
\qquad\qquad\qquad 9s^5 \qquad\;\; +36s \\
\hline
\qquad\qquad\qquad\qquad 9s^4+36 \\
\qquad\qquad\qquad\qquad 9s^4+36 \\
\hline
\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation.

$$\therefore s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$$

$$\underbrace{(s^4+4)}_{\text{Even polynomial}} \quad \underbrace{(s^3+5s^2+9s+9)}_{\text{Quotient polynomial}}=0$$

The routh array is constructed for quotient polynomial as shown below.

s^3 : 1 9

s^2 : 5 9

s^1 : 7.2

s^0 : 9

Column-1

$$s^1 : \frac{5\times 9-9\times 1}{5}$$

s^1 : 7.2

$$s^0 : \frac{7.2\times 9-0\times 5}{7.2}$$

s^0 : 9

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on the left half of s-plane.

To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary polynomial is, $s^4+4=0$.

Put, $s^2=x$ in the auxiliary equation, $\therefore s^4+4=x^2+4=0$

$$\therefore x^2=-4 \quad \Rightarrow \quad x=\pm\sqrt{-4}=\pm j2=2\angle 90^\circ \text{ or } 2\angle -90^\circ$$

$$\text{But, } s=\pm\sqrt{x}=\pm\sqrt{2\angle 90^\circ} \text{ or } \pm\sqrt{2\angle -90^\circ} = \pm\sqrt{2}\angle 90^\circ/2 \text{ or } \pm\sqrt{2}\angle -90^\circ/2$$

$$=\pm\sqrt{2}\angle 45^\circ \text{ or } \pm\sqrt{2}\angle -45 = \pm(1+j1) \text{ or } \pm(1-j1)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by roots of quotient polynomial and auxiliary polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

(a) The system is unstable.

(b) Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

EXAMPLE 4.7

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

SOLUTION

The characteristic equation of the system is, $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5	:	1	8	7	.... Row-1
s^4	:	4	8	4	.... Row-2

Divide s^4 row by 4 to simplify the calculations.

s^5	:	1	8	7	.... Row-1
s^4	:	1	2	1	.... Row-2
s^3	:	1	1		.... Row-3
s^2	:	1	1		.... Row-4
s^1	:	$\in$			.... Row-5
s^0	:	1			.... Row-6

Column-1

$s^3 : \dfrac{1\times 8 - 2\times 1}{1} \quad \dfrac{1\times 7 - 1\times 1}{1}$

$s^3 : 6 \quad 6$

Divide by 6

$s^3 : 1 \quad 1$

$s^2 : \dfrac{1\times 2 - 1\times 1}{1} \quad \dfrac{1\times 1 - 0\times 1}{1}$

$s^2 : 1 \quad 1$

$s^1 : \dfrac{1\times 1 - 1\times 1}{1}$

$s^1 : 0$

Let $0 \to \in$

$s^1 : \in$

$s^0 : \dfrac{\in\times 1 - 0\times 1}{\in}$

$s^0 : 1$

When $\in \to 0$, there is no sign change in the first column of routh array. But we have a row of all zeros (s^1 row or row-5) and so there is a possibility of roots on imaginary axis. This can be found from the roots of auxiliary polynomial. Here the auxiliary polynomial is given by s^2 row.

The auxiliary polynomial is, $s^2 + 1 = 0$; $\quad \therefore\ s^2 = -1$ or $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are +j1and –j1, lying on imaginary axis. The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots of characteristic equation are lying on imaginary axis and so the system is limitedly or marginally stable. The remaining three roots of characteristic equation are lying on the left half of s-plane.

RESULT

(a) The system is limitedly or marginally stable.

(b) Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

EXAMPLE 4.8

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

SOLUTION

The characteristic polynomial of the system is, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

The given characteristic polynomial is 6^{th} order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s as shown below.

$$
\begin{array}{lclllll}
s^6 & : & 1 & 3 & 3 & 1 & \text{.... Row-1} \\
s^5 & : & 1 & 3 & 2 & & \text{.... Row-2} \\
s^4 & : & \in & 1 & 1 & & \text{.... Row-3} \\
s^3 & : & \dfrac{3\in-1}{\in} & \dfrac{2\in-1}{\in} & & & \text{.... Row-4} \\
s^2 & : & \dfrac{-2\in^2+4\in-1}{3\in-1} & 1 & & & \text{.... Row-5} \\
s^1 & : & \dfrac{4\in^2-\in}{2\in^2-4\in+1} & & & & \text{.... Row-6} \\
s^0 & : & 1 & & & & \text{.... Row-7}
\end{array}
$$

$$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^4 : 0 \quad 1 \quad 1$$

let $0 \rightarrow \in$

$$s^4 : \in \quad 1 \quad 1$$

$$s^3 : \frac{\in\times3-1\times1}{\in} \quad \frac{\in\times2-1\times1}{\in}$$

$$s^3 : \frac{3\in-1}{\in} \quad \frac{2\in-1}{\in}$$

$$s^2 : \frac{\dfrac{3\in-1}{\in}-\dfrac{2\in-1}{\in}\times\in}{\dfrac{3\in-1}{\in}} \quad \frac{\dfrac{3\in-1}{\in}\times1-0\times\in}{\dfrac{3\in-1}{\in}}$$

$$s^2 : \frac{-2\in^2+4\in-1}{3\in-1} \quad 1$$

$$s^1 : \frac{\dfrac{-2\in^2+4\in-1}{3\in-1}\times\dfrac{2\in-1}{\in}-\dfrac{3\in-1}{\in}\times1}{\dfrac{-2\in^2+4\in-1}{3\in-1}}$$

$$s^1 : \frac{(-2\in^2+4\in-1)(2\in-1)-(3\in-1)(3\in-1)}{\in(-2\in^2+4\in-1)}$$

$$s^1 : \frac{-4\in^3+\in^2}{\in(-2\in^2+4\in-1)}=\frac{4\in^2-\in}{2\in^2-4\in+1}$$

$$s^0 : \frac{\dfrac{4\in^2-\in}{4\in^2-4\in+1}\times1-0\times\dfrac{-2\in^2+4\in-1}{3\in-1}}{(4\in^2-\in)/(4\in^2-4\in+1)}$$

$$s^0 : 1$$

On letting $\in \rightarrow 0$, we get,

$$
\begin{array}{lclllll}
s^6 & : & 1 & 3 & 3 & 1 & \text{.... Row-1} \\
s^5 & : & 1 & 3 & 2 & & \text{.... Row-2} \\
s^4 & : & 0 & 1 & 1 & & \text{.... Row-3} \\
s^3 & : & -\infty & -\infty & & & \text{.... Row-4} \\
s^2 & : & 1 & 1 & & & \text{.... Row-5} \\
s^1 & : & 0 & & & & \text{.... Row-6} \\
s^0 & : & 1 & & & & \text{.... Row-7}
\end{array}
$$

+ to −

− to +

Since there is a row of all zeros (s^1 row) there is a possibility of roots on imaginary axis. The auxiliary polynomial is $s^2 + 1 = 0$.

The roots of auxiliary polynomial are, $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots are lying on imaginary axis. Therefore divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

The characteristic polynomial can be expressed as a product of auxiliary polynomial and quotient polynomial.

$$\therefore s^6+s^5+3s^4+3s^3+3s^2+2s+1=0 \Rightarrow \underbrace{(s^2+1)}_{\text{Even polynomial}}\ \underbrace{(s^4+s^3+2s^2+2s+1)}_{\text{Quotient polynomial}}=0$$

The routh array for quotient polynomial is constructed as shown below.

$$
\begin{array}{lcllll}
s^4 & : & 1 & 2 & 1 & \text{.... Row-1} \\
s^3 & : & 1 & 2 & & \text{.... Row-2} \\
s^2 & : & \in & 1 & & \text{.... Row-3} \\
s^1 & : & \dfrac{2\in-1}{\in} & & & \text{.... Row-4} \\
s^0 & : & 1 & & & \text{.... Row-5}
\end{array}
$$

$$s^2 : \frac{1\times2-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 0 \quad 1$$

let $0 \rightarrow \in$

$$s^2 : \in \quad 1$$

$$s^1 : \frac{\in\times2-1\times1}{\in}$$

$$s^1 : \frac{2\in-1}{\in}$$

$$s^0 : \frac{\dfrac{2\in-1}{\in}\times1-0\times\in}{(2\in-1)/\in}$$

$$s^0 : 1$$

On letting $\in \to 0$, we get

s^4	:	1	2	1	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	0	1		.... Row-3
s^1	:	$-\infty$			.... Row-4
s^0	:	1			.... Row-5

+ to −

− to +

Column-1

$$
\begin{array}{r|l}
 & s^4+s^3+2s^2+2s+1 \quad \text{(Quotient polynomial)} \\ \hline
s^2+1 & s^6+s^5+3s^4+3s^3+3s^2+2s+1 \\
\text{(Even polynomial)} & \underline{\overset{(-)}{s^6} \quad \overset{(-)}{+} s^4} \\
 & s^5+2s^4+3s^3+3s^2+2s+1 \\
 & \underline{\overset{(-)}{s^5} \quad \overset{(-)}{+} s^3} \\
 & 2s^4+2s^3+3s^2+2s+1 \\
 & \underline{(-)2s^4 \quad (-)+2s^2} \\
 & 2s^3+s^2+2s+1 \\
 & \underline{(-)2s^3 \quad (-)+2s} \\
 & s^2 \quad +1 \\
 & \underline{(-)s^2 \quad (-)+1} \\
 & 0
\end{array}
$$

On examining the first column of the routh array of quotient polynomial, we found that there are two sign changes. Hence two roots are lying on the right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

The roots of characteristic equation are given by roots of auxiliary polynomial and quotient polynomial. Hence two roots are lying on imaginary axis, two roots are lying on right half of s-plane and the remaining two roots are lying on left half of s-plane. Hence the system is unstable.

RESULT

(a) The system is unstable.

(b) Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.9

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s)=\dfrac{K}{s(s+1)\,(s+2)}$.

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)}=\dfrac{G(s)}{1+G(s)}=\dfrac{\dfrac{K}{s(s+1)\,(s+2)}}{1+\dfrac{K}{s(s+1)\,(s+2)}}=\dfrac{K}{s(s+1)\,(s+2)+K}$

The characteristic equation is, $\quad s\,(s+1)\,(s+2)+K=0$

$$\therefore s\,(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\dfrac{6-K}{3}$	
s^0	:	K	

Column-1

$$s^1 : \frac{3\times 2-K\times 1}{3}$$

$$s^1 : \frac{6-K}{3}$$

$$s^0 : \frac{\dfrac{6-K}{3}\times K-0\times 3}{(6-K)/3}$$

$$s^0 : K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

EXAMPLE 4.10

The open loop transfer function of a unity feedback control system is given by,

$$G(s) = \frac{K}{(s+2)(s+4)(s^2+6s+25)}.$$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}} = \frac{K}{(s+2)(s+4)(s^2+6s+25)+K}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is, $(s+2)(s+4)(s^2+6s+25)+K=0$.

$$\therefore (s^2+6s+8)(s^2+6s+25)+K=0 \quad \Rightarrow \quad s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$\begin{array}{lclll} s^4 & : & 1 & 69 & 200+K \text{..... Row-1} \\ s^3 & : & 12 & 198 & \text{..... Row-2} \end{array}$$

Divide s^3 row by 12 to simplify the calculations

$$\begin{array}{lcclll} s^4 & : & 1 & 69 & 200+K & \text{..... Row-1} \\ s^3 & : & 1 & 16.5 & & \text{..... Row-2} \\ s^2 & : & 52.5 & 200+K & & \text{..... Row-3} \\ s^1 & : & \dfrac{666.25-K}{52.5} & & & \text{..... Row-4} \\ s^0 & : & 200+K & & & \text{..... Row-5} \end{array}$$

Column-1

$$s^2 : \frac{1\times 69-16.5\times 1}{1} \quad \frac{1\times(200+K)}{1}$$

$$s^2 : 52.5 \qquad 200+K$$

$$s^1 : \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$$

$$s^1 : \frac{666.25-K}{52.5}$$

$$s^0 : \frac{\dfrac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$$

$$s^0 : 200+K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, (666.25-K) > 0.

Since (666.25-K) > 0, should be less than 666.25.

From s^0 row, for the system to be stable, (200+K) > 0

Since (200+K) > 0, K should be greater than -200, but practical values of K starts from 0. Hence K should be greater than zero.

$\therefore$ The range of K for the system to be stable is 0<K<666.25.

When K= 666.25 the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When K=666.25, the coefficients of auxiliary equation are given by the s^2 row.

$\therefore$ The auxiliary equation is, $52.5s^2+200+K=0$

$$52.5s^2 + 200 + 666.25 = 0$$

$$s^2 = \frac{-200 - 666.25}{52.5} = -16.5$$

$$s = \pm\sqrt{-16.5} = \pm j\sqrt{16.5} = \pm j4.06$$

When K = 666.25, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

$\therefore$ The frequency of oscillation, $\omega = 4.06$ rad/sec.

RESULT

(a) The range of K for stability is 0<K<666.25

(b) The system oscillates when K = 666.25

(c) The frequency of oscillation, $\omega = 4.06$ rad/sec. (When K = 666.25)

EXAMPLE 4.11

The open loop transfer function of a unity feedback system is given by, $G(s) = \dfrac{K(s+1)}{s^3 + as^2 + 2s + 1}$. Determine the value of K and a so that the system oscillates at a frequency of 2 rad/sec.

SOLUTION

$$\left.\begin{array}{r}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K(s+1)}{s^3+as^2+2s+1}}{1+\dfrac{K(s+1)}{s^3+as^2+2s+1}} = \frac{K(s+1)}{s^3+as^2+2s+1+K(s+1)}$$

The characteristic equation is, $s^3 + as^2 + 2s + 1 + K(s+1) = 0$.

$$s^3 + as^2 + 2s + 1 + Ks + K = 0 \quad \Rightarrow \quad s^3 + as^2 + (2+K)s + 1 + K = 0$$

The routh array of characteristic polynomial is constructed as shown below. The maximum power of s is odd, hence the first row of routh array is formed using coefficients of odd powers of s and the second row of routh array is formed using coefficients of even powers of s.

If the elements of s^1 row are all zeros then there exists an even polynomial (or auxiliary polynomial). If the roots of the auxiliary polynomial are purely imaginary then the roots are lying on imaginary axis and the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

Routh array

s^3 : 1 2+K

s^2 : a 1+K

s^1 : $\dfrac{a(2+K)-(1+K)}{a}$

s^0 : 1+K

From s^2 row, the auxiliary polynomial is,

$$as^2+(1+K)=0 \quad \Rightarrow \quad as^2=-(1+K) \quad \Rightarrow \quad s=\pm j\sqrt{\frac{1+K}{a}}$$

Given that, $s=\pm j2$, $\quad \therefore \sqrt{\dfrac{1+K}{a}}=2 \quad \Rightarrow \quad \dfrac{1+K}{a}=4 \quad \Rightarrow \quad K=4a-1$

From s^1 row, $\dfrac{a(2+K)-(1+K)}{a}=0 \Rightarrow a(2+K)-(1+K)=0 \Rightarrow 2a+Ka-1-K=0$

$$\therefore \quad 2a-1+K(a-1)=0$$

Put, $K = 4a - 1$

$$\therefore 2a-1+(4a-1)(a-1)=0 \quad \Rightarrow \quad 2a-1+4a^2-4a-a+1=0 \quad \Rightarrow \quad 4a^2-3a=0 \text{ (or) } a(4a-3)=0$$

Since $a \neq 0$, $\quad 4a-3=0$, $\quad \therefore a=3/4$

When $a=(3/4)$, $\quad K=4a-1=4\times(3/4)-1=2$

RESULT

When the system oscillates at a frequency of 2 rad/sec, $K=2$ and $a=3/4$.

EXAMPLE 4.12

A feedback system has open loop transfer function of $G(s)=\dfrac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of K for stability of closed loop system.

SOLUTION

Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by, $1-s$, (i.e., $e^{-sT} \approx 1-sT$).

$$\therefore G(s)=\frac{Ke^{-s}}{s(s^2+5s+9)}\approx\frac{K(1-s)}{s(s^2+5s+9)}$$

The closed loop transfer function $\Big\}$ $\dfrac{C(s)}{R(s)}=\dfrac{G(s)}{1+G(s)}=\dfrac{\dfrac{K(1-s)}{s(s^2+5s+9)}}{1+\dfrac{K(1-s)}{s(s^2+5s+9)}}=\dfrac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is, $s(s^2+5s+9)+K(1-s)=0$

$$\therefore s(s^2+5s+9)+K(1-s)=s^3+5s^2+9s+K-Ks=0 \Rightarrow s^3+5s^2+(9-K)s+K=0$$

The routh array of characteristic polynomial is constructed as shown below.

The maximum power of s in the characteristic polynomial is odd, hence form the first row of routh array using coefficients of odd powers of s and second row of routh array using coefficients of even powers of s.

$$\begin{array}{lll} s^3 & : & 1 \quad\quad 9-K \\ s^2 & : & 5 \quad\quad K \\ s^1 & : & 9-1.2K \\ s^0 & : & K \end{array}$$

$s^1 : \dfrac{5\times(9-K)-K\times 1}{5}$

$s^1 : \dfrac{45-5K-K}{5}$

$s^1 : \dfrac{45-6K}{5} \approx 9-1.2K$

$s^0 : \dfrac{(9-1.2K)\times K}{(9-1.2K)}$

$s^0 : K$

From s^1 row, for stability of the system, $(9-1.2K)>0$

If $(9-1.2K)>0$ then $1.2K<9$; $\therefore K<\dfrac{9}{1.2}=7.5$

From s^0 row, for stability of the system, $K>0$

Finally we can conclude that for stability of the system K should be in the range of $0<K<7.5$

RESULT

For stability of the system K should be in the range of, $0<K<7.5$.

4.4 MATHEMATICAL PRELIMINARIES FOR NYQUIST STABILITY CRITERION

Let F(s) be a function of s, which is expressed as a ratio of two polynomials in s, as shown in equation (4.14), (the polynomials are expressed in the factored form).

$$F(s)=\frac{(s-z_1)\,(s-z_2)\,......\,(s-z_m)}{(s-p_1)\,(s-p_2)\,......\,(s-p_n)} \qquad .....(4.14)$$

The roots of numerator polynomial are zeros and the roots of denominator polynomial are poles. The function has m number of zeros and n number of poles.

Here, s is a complex variable expressed as, $s=\sigma+j\omega$, where s is real part of σ and ω is imaginary part of s. (The s is also called complex frequency). For a particular value of σ and ω, the s will represent a point in the s-plane.

Since s is a complex variable, the function F(s) will also be a complex quantity for any value of s. Hence, F(s) can also be expressed as, F(s) = u+jv, where u is real part of F(s) and v is imaginary part of F(s). Let us define another complex plane called F(s)-plane, with coordinates u and v. For a particular value of s, the F(s) will represent a point in F(s)-plane.

Therefore, for every point s in the s-plane at which F(s) is analytic, there exists a corresponding point F(s) in the F(s)-plane. Hence it can be concluded that the function F(s) maps the points in the s-plane into the F(s)-plane.

> *Note : A function is analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function (or it derivatives) does not exist are called singular points.*

Since any number of points of analycity in the s-plane can be mapped into the F(s)- plane it can be concluded that for a contour in the s-plane which does not go through any singular point, there exists a corresponding contour in the F(s)-plane as shown in fig 4.2.

The table 4.2 shows examples of arbitrary s-plane contours and their corresponding F(s)-plane contours (exact shape is not shown).

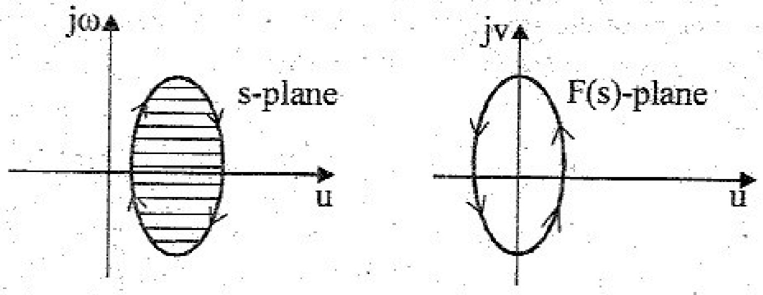


Fig 4.2 : *An arbitrary contour in s-plane and its corresponding contour in F(s)-plane*

Normally the direction of arbitrary contour in s-plane is chosen as clockwise. Here zeros are marked by small circles (o) and poles by (X).

On observing the s-plane contours and the corresponding F(s)-plane contours shown in table-4.2, it can be proved that there exists a relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)- plane by the corresponding F(s)-plane contour.

> ***Note :*** *For the development of Nyquist criterion, the exact shape of the contour is not required but only the number of encirclements of the origin of the F(s) - plane is essential.*

Concept of encircled and enclosed

It is important to distinguish between the concept of encircled and enclosed which are frequently used to apply Nyquist criterion.

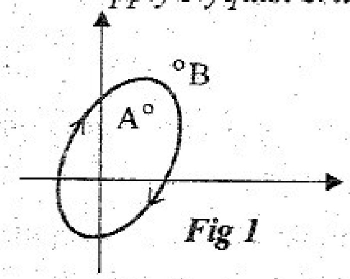


Fig 1

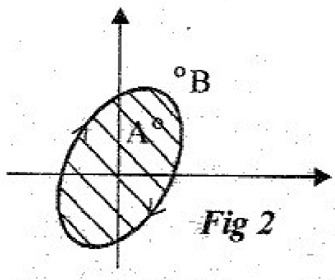


Fig 2

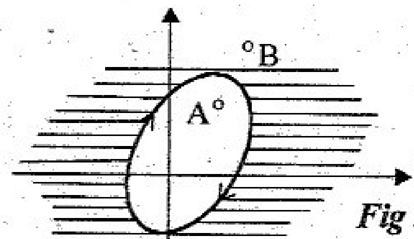
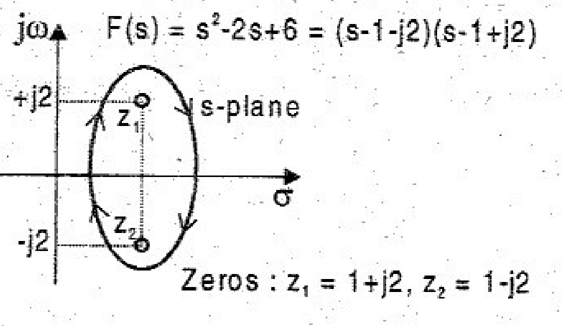
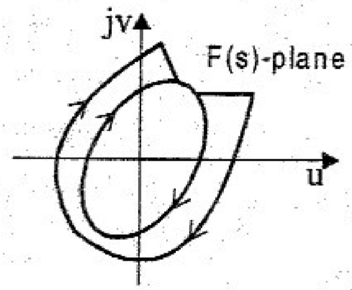


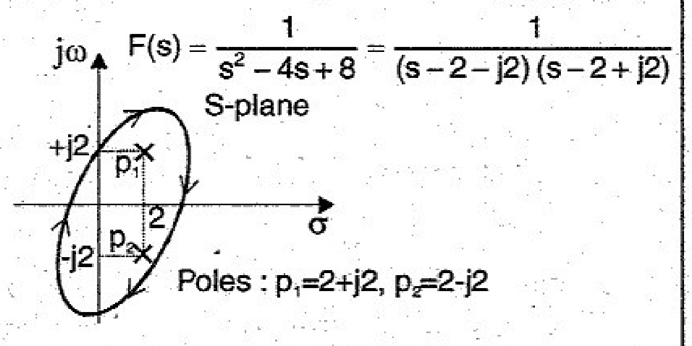
Fig 3

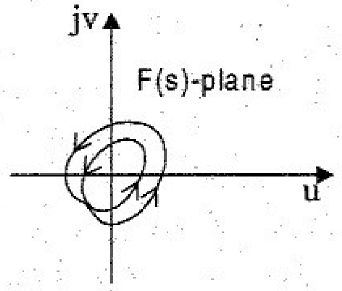
Encircled : *A point is said to be encircled by a closed path if it is found inside the path. With reference to fig 1, the point A is encircled in the clockwise direction and the point B is not encircled.*

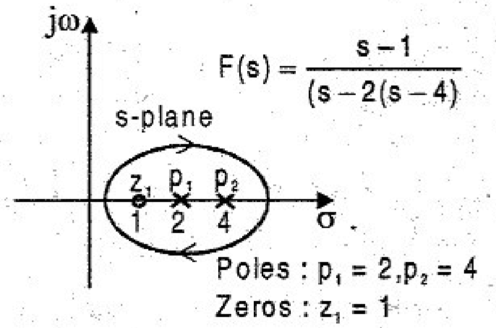
Enclosed : *Any point or region is said to be enclosed by a closed path, if it is found to lie to the right of the path when the path is traversed in the prescribed direction. The shaded regions in fig 2 and 3 are the regions enclosed by the closed path. With reference to fig 2, the point A is enclosed by closed path and the point B is not enclosed. With reference to fig 3 the point A is not enclosed by closed path but point B is enclosed.*

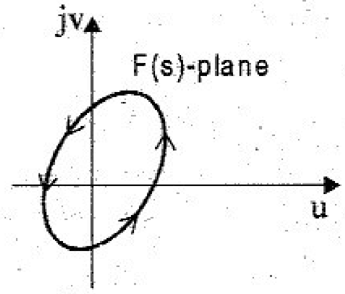
TABLE-4.2

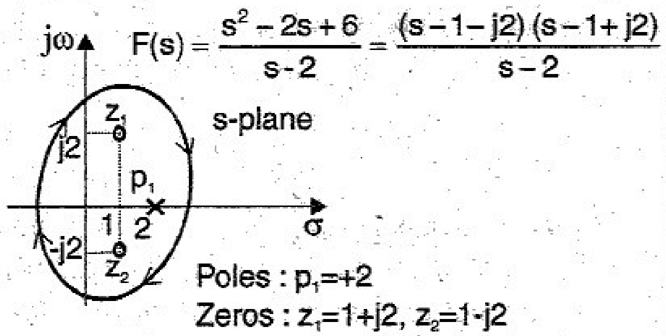
The function, F(s) and s-plane contour	**F(s)-plane contour**
	

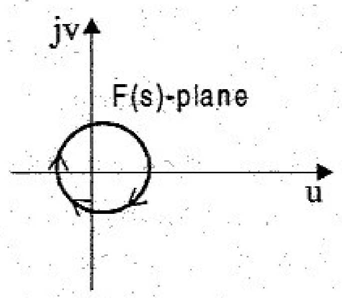


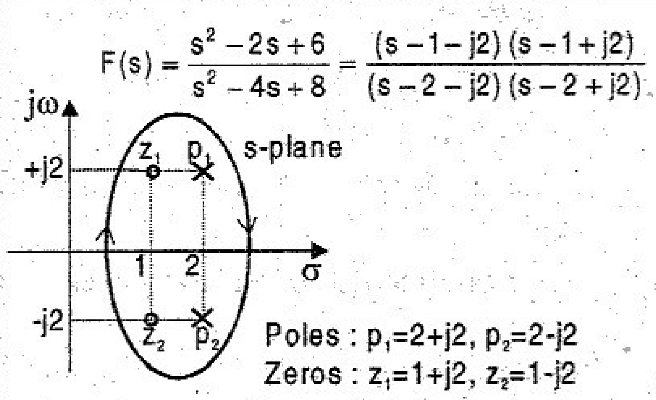


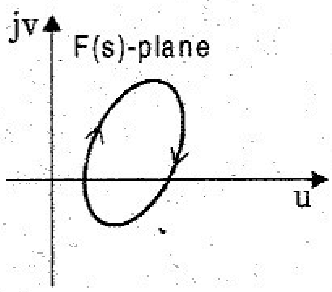


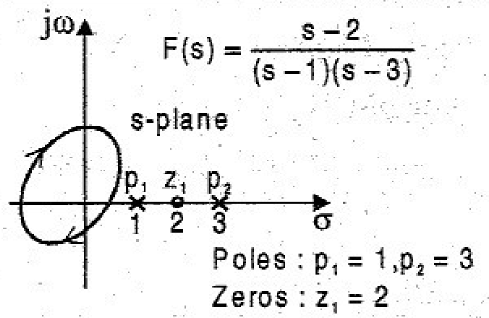


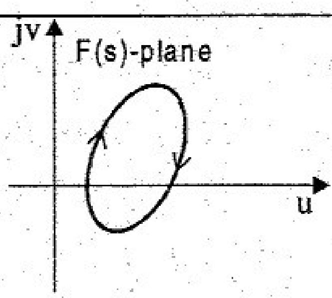












The summary of relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour, are given below.

1. If s-plane closed contour encloses Z number of zeros in the right half of s-plane then the corresponding contour in F(s)-plane will encircle, the origin of F(s)-plane Z times in the clockwise direction.
2. If s-plane closed contour encloses P number of poles in the right half of s-plane then the corresponding contour in F(s)-plane will encircle the origin of F(s)- plane P times in anticlockwise direction.
3. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P > Z$, then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane $(P-Z)$ times in the anti-clockwise direction.
4. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P < Z$ then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane $(Z-P)$ times in the clockwise direction.
5. If s-plane closed contour encloses Z zeros and P poles in right half of s-plane and if $P = Z$, then corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.
6. If the s-plane closed contour does not enclose any pole or zero, then the corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.

The relation between the enclosure of poles and zeros of F(s) lying on the right half of s-plane by the s-plane contour and the encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour is called ***principle of argument.***

The principle of argument is stated as follows.

Let F(s) is a single valued rational function and is analytic in a given region in the s-plane except at some points. Now, if an arbitrary closed contour is chosen in the s-plane, so that F(s) is analytic at every point on the closed contour in s-plane then the corresponding F(s)-plane contour mapped in the F(s)-plane will encircle the origin N times in anticlockwise direction where N is the difference between the number of poles and number of zeros of F(s) that are encircled by the chosen closed contour in s-plane.

Mathematically, it can be expressed as, $N = P-Z$.

where, N = Number of encirclement of origin of F(s)-plane, made by F(s)-contour.

Z = Number of zeros of F(s) lying on right half of s-plane and enclosed by the s-plane closed contour.

P = Number of poles of F(s) lying on right half of s-plane and enclosed by the s-plane closed contour.

The value of N can be positive, zero or negative. Based on the sign of N, following conclusions can be made, provided the arbitrary s-plane contour is chosen in the clockwise direction.

1. If N is positive, then direction of encirclement of origin of F(s)-plane will be anticlockwise.

2. If N is zero, then there will be no encirclement of origin of F(s)-plane.

3. If N is negative, then direction of encirclement of origin of F(s)-plane will be clockwise.

4.5 NYQUIST STABILITY CRITERION

Consider the closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

The characteristic equation of the system is given by the condition, $1+ G(s)\,H(s)=0$.

Let, F(s) = 1 + G(s)H(s).

The loop transfer function G(s)H(s) can be expressed as,

$$G(s)H(s) = \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}, \text{ where } m \leq n \qquad .....(4.15)$$

$$\therefore F(s) = 1 + G(s)H(s) = 1 + \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$= \frac{(s+p_1)\,(s+p_2)\,.....\,(s+p_n) + K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.\,.\,(s+p_n)}$$

$$= \frac{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)} \qquad .....(4.16)$$

In equation (4.16), z_1', z_2' ..... z_n', are zeros of F(s), which are obtained by combining the numerator and denominator polynomial of G(s)H(s).

For the condition F(s) = 0, the numerator of F(s) should be equal to zero.

$$\therefore (s + z_1')\,(s + z_2')\,.......\,(s + z_n') = 0 \qquad .....(4.17)$$

We can say that equation (4.17) is the characteristic equation of the system. For the stability of the system the roots of the characteristic equation should not lie on the right half s-plane. The roots of characteristic equation are zeros of F(s) and also they are poles of closed loop transfer function.

Hence we can conclude that for the stability of closed loop system the zeros of F(s) should not lie on the right half s-plane.

***Note :** For a unity feedback system.*

$$G(s)H(s) = G(s) = \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}{1+\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}$$

$$= \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n) + K(s+z_1)\,(s+z_2)\,.....(s+z_m)}$$

$$= \frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}$$

From the above equation we can say that the poles of closed loop transfer function are z_1', z_2', ... z_n'.

Let us choose an arbitrary contour in the s-plane which encircles the right half zeros and poles of F(s) (equation (4.16)). The principle of argument (explained in section 4.4) states that the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane, N times in the anticlockwise direction.

Let, N = Number of anticlockwise encirclement

Now, N = P – Z .....(4.18)

where, P = Number of poles of F(s) (or poles of loop transfer function) lying on right half s-plane

Z = Number of zeros of F(s) (or poles of closed loop transfer function) lying on right half s-plane

> ***Note :*** *The stability is related to poles lying on right half s-plane and so, while applying principle of argument only poles and zeros lying on right half s-plane alone are considered.*

For the stability of the system the roots of characteristic equation and so the zeros of F(s) should not lie on the right half of s-plane. Hence for a stable system Z = 0. Hence from equation (4.18) we get,

When $Z = 0$, $\quad N = P$ .....(4.19)

When $Z \neq 0$, $\quad N \neq P$ .....(4.20)

From equation (4.15) and (4.16) we can say that the poles of F(s) are also poles of loop transfer function. Hence for the stability of the system, (with reference to equation (4.19) and equation (4.20)) number of poles of loop transfer function lying on right of s-plane should be equal to anticlockwise encirclement of the origin of F(s)-plane. If this condition is not met the system is unstable.

> The principle of argument can also be used to find the number of poles of closed loop transfer function lying on right half of s-plane.
>
> Let, $\quad$ M = Number of clockwise encirclement
>
> Now, $M = Z - P$
>
> When $P = 0$, $\quad M = Z$
>
> Therefore, when there is no right half open loop poles, number of clockwise encirclement of origin of F(s)-plane gives number of poles of closed loop transfer function lying on right half s-plane.

The loop transfer function, G(s)H(s) can be expressed as,

$$G(s)H(s) = [1+G(s)\,H(s)]-1 = F(s) - 1 \quad .....(4.21)$$

From equation (4.21) it can be concluded that the contour of F(s) drawn with respect to origin of F(s)-plane is same as the contour of F(s)–1 drawn with respect to $-1+j0$ of F(s)-plane as shown in fig 4.3.

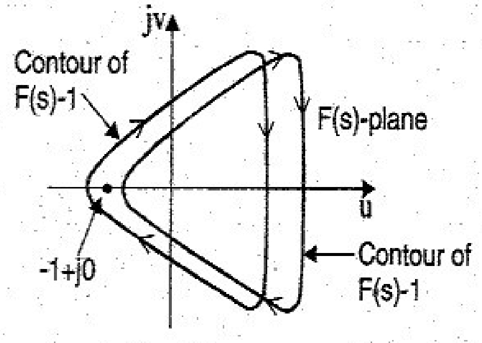


Fig 4.3

Thus the encirclement of the origin of F(s)-plane by the contour of F(s) is equivalent to the encirclement of the point $-1 + j0$ by the contour of F(s) – 1.

From equation (4.21), we can say that F(s) – 1, represents loop transfer function G(s)H(s). Hence contour of F(s) – 1 is same as contour of G(s)H(s), and F(s)-plane is G(s)H(s)-plane.

Therefore, the encirclement of $-1 + j0$ point of G(s)H(s)-contour in the G(s)H(s)-plane can be used to determine the stability of closed loop system. The Nyquist stability criterion have been proposed based on this concept.

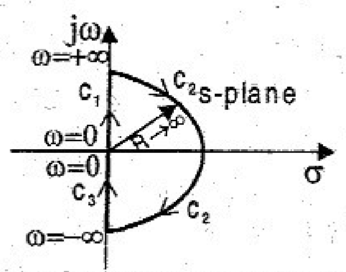


Fig 4.4 : *Nyquist Contour*

In order to investigate the presence of poles of G(s)H(s) on the right half s-plane a contour, C is chosen such that it encloses the entire right half s-plane as shown in fig 4.4, such a contour C is called ***Nyquist contour***.

The Nyquist contour is directed clockwise and comprises of three segments,

1. An infinite line segment C_1 along the positive imaginary axis.
2. An arc, C_2 of infinite radius, enclosing the entire right half of s-plane.
3. An infinite line segment C_3 along the negative imaginary axis.

Along C_1, $s = j\omega$, with ω varying from 0 to $+\infty$.

Along C_2, $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$, with θ varying from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

Along C_3, $s = j\omega$, with ω varying from $-\infty$ to 0.

Using the loop transfer function G(s)H(s), the Nyquist contour-C of s-plane, is mapped to G(s)H(s)-plane. The mapped contour in G(s)H(s)-plane is called G(s)H(s)-contour.

> ***Note :*** *The s-plane is a complex plane. Any point on a complex plane can be expressed by the complex number in polar form, $Re^{j\theta}$, where R is the magnitude and θ is the argument (or phase).*

Now the Nyquist stability criterion can be stated as follows.

"If the G(s)H(s) contour in the G(s)H(s)-plane corresponding to Nyquist contour in the s-plane encircles the point -1 + j0 in the anticlockwise direction as many times as the number of right half s-plane poles of G(s)H(s), then the closed loop system is stable".

In examining the stability of linear control systems using the Nyquist stability criterion, we come across the following three situations.

1. **No encirclement of –1 + j0 point :** This implies that the system is stable if there are no poles of G(s)H(s) in the right half s-plane. If there are poles on right half s-plane then the system is unstable.

2. **Anticlockwise encirclements of –1 + j0 point :** In this case the system is stable if the number of anticlockwise encirclements is same as the number of poles of G(s)H(s) in the right half s-plane. If the number of anticlockwise encirclements is not equal to number of poles on right half s-plane then the system is unstable.

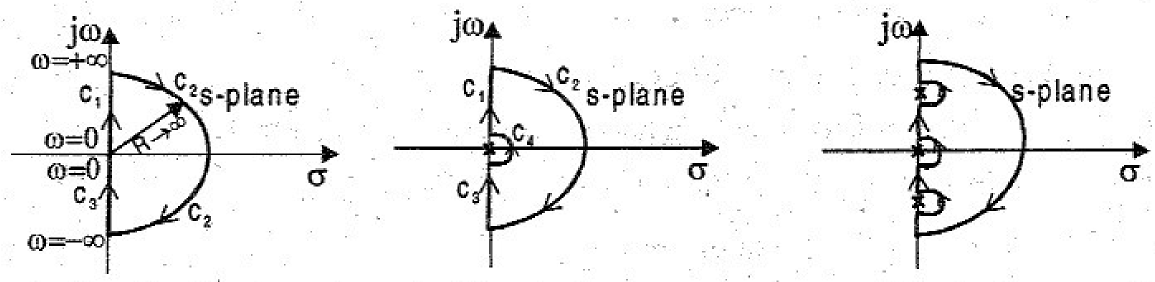
3. **Clockwise encirclements of the –1 + j0 point :** In this case the system is always unstable. Also in this case, if no poles of G(s)H(s) in right half s-plane, then the number of clockwise encirclement is equal to number of poles of closed loop system on right half s-plane.

PROCEDURE FOR INVESTIGATING THE STABILITY USING NYQUIST CRITERION

The following procedure can be followed to investigate the stability of closed loop system from the knowledge of open loop system, using Nyquist stability criterion.

1. Choose a Nyquist contour as shown in fig 4.5, which encloses the entire right half s-plane except the singular points. The Nyquist contour encloses all the right half s-plane poles and zeros of G(s)H(s). [The poles on imaginary axis are singular points and so they are avoided by taking a detour around it as shown in fig 4.5 b and c].

> ***Note :*** *For mapping a contour from s-plane to G(s)H(s) plane the Nyquist contour in s-plane should be analytic at every point. At singular points it is not analytic.*



a. *Nyquist Contour when there is no pole on imaginary axis*
b. *Nyquist Contour when there are poles at origin*
c. *Nyquist Contour when there are poles on imaginary axis and at origin*

Fig 4.5 : *Nyquist Contour*

2. The Nyquist contour should be mapped in the G(s)H(s)-plane using the function G(s)H(s) to determine the encirclement $-1+j0$ point in the G(s)H(s)-plane. The Nyquist contour of fig 4.5b can be divided into four sections C_1, C_2, C_3 and C_4. The mapping of the four sections in the G(s)H(s)-plane can be carried sectionwise and then combined together to get entire G(s)H(s)-contour.

3. In section C_1 the value of ω varies from 0 to $+\infty$. The mapping of section C_1 is obtained by letting $s = j\omega$ in G(s)H(s) and varying ω from 0 to $+\infty$,

$$\text{i.e. } G(s)H(s)\Big|_{\substack{s=j\omega \\ \omega=0 \text{ to } \infty}} = G(j\omega)\,H(j\omega)\Big|_{\omega=0 \text{ to } \infty}$$

The locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to $+\infty$ will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C_1 in s-plane. This locus is the polar plot of $G(j\omega)H(j\omega)$. There are three ways of mapping this section of G(s)H(s)-contour, they are,

(i) Calculate the values of $G(j\omega)H(j\omega)$ for various values of ω and sketch the actual locus of $G(j\omega)H(j\omega)$.

(or)

(ii) Separate the real part and imaginary part of $G(j\omega)H(j\omega)$. Equate the imaginary part to zero, to find the frequency at which the $G(j\omega)H(j\omega)$ locus crosses real axis (to find phase crossover frequency). Substitute this frequency on real part and find the crossing point of the locus on real axis. Sketch the approximate locus of $G(j\omega)H(j\omega)$ from the knowledge of type number and order of the system (or from the value of $G(j\omega)H(j\omega)$ at $\omega = 0$ and $\omega = \infty$).

(iii) Separate the magnitude and phase of $G(j\omega)H(j\omega)$. Equate the phase of $G(j\omega)H(j\omega)$ to -180^0 and solve for ω. This value of ω is the phase crossover frequency and the magnitude at this frequency is the crossing point on real axis. Sketch the approximate root locus as mentioned in method (ii).

4. The section C_2 of Nyquist contour has a semicircle of infinite radius. Therefore, every point on section C_2 has infinite magnitude but the argument varies from $+\pi/2$ to $-\pi/2$. Hence the mapping of section C_2 from s-plane to G(s)H(s) plane can be obtained by letting $s = \underset{R\to\infty}{Lt}\; Re^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$.

Consider the loop transfer function in time constant form and with y number of poles at origin, as shown below.

$$G(s)H(s) = \frac{K\,(1+sT_1)\,(1+sT_2)\,(1+sT_3)......}{s^y\,(1+sT_a)\,(1+sT_b)\,(1+sT_c)......}$$

Let G(s)H(s) has m zeros & n poles including poles at origin. For practical systems, n >m.

Since, $s \to Re^{j\theta}$ and $R \to \infty$, the term (1+sT) can be approximated to sT, [i.e., $(1+sT) \approx sT$].

$$\therefore G(s)H(s) \approx K\frac{sT_1 \times sT_2 \times sT_3.....}{s^y \times sT_a \times sT_b \times sT_c.....} = K_1\frac{s^m}{s^n} = \frac{K_1}{s^{n-m}}$$

On letting, $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$ we get,

$$G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{Lt}\ Re^{j\theta}} = \frac{K_1}{\underset{R\to\infty}{Lt}\ (Re^{j\theta})^{n-m}} = 0e^{-j\theta(n-m)}$$

When $\theta = \frac{\pi}{2}$, $G(s)H(s) = 0e^{-j\frac{\pi}{2}(n-m)}$

When $\theta = -\frac{\pi}{2}$, $G(s)H(s) = 0e^{+j\frac{\pi}{2}(n-m)}$

From the above two equations we can conclude that the section C_2 of Nyquist contour in s-plane is mapped as circles/circular arc around origin with radius tending to zero in the G(s)H(s)-plane.

5. In section C_3, the value of ω varies from $-\infty$ to 0. The mapping of section C_3 is obtained by letting $s = +j\omega$ in G(s)H(s) and varying ω from $-\infty$ to 0.

$$\text{i.e., } G(s)H(s)\Bigg|_{\substack{s=+j\omega \\ \omega=-\infty \text{ to } 0}} = G(j\omega)H(j\omega)\Bigg|_{\omega=-\infty \text{ to } 0.}$$

The locus of G(jω)H(jω) as ω is varied from $-\infty$ to 0 will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C_3 in s-plane. This locus is the inverse polar plot of G(jω)H(jω). The inverse polar plot is given by the mirror image of polar plot with respect to real axis.

6. The section C_4 of Nyquist contour has a semicircle of zero radius. Therefore every point on semicircle has zero magnitude but the argument varies from $-\pi/2$ to $+\pi/2$. Hence the mapping of section C_4 from s-plane to G(s)H(s)-plane can be obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{-j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$.

Consider the loop transfer function in time constant form and with y number of poles at origin as shown below.

$$G(s)H(s) = \frac{K\,(1+sT_1)\,(1+sT_2)\,(1+sT_3)\ .....}{s^y\,(1+sT_a)\,(1+sT_b)\,(1+sT_c)\ .....}$$

Let G(s)H(s) has m zeros & n poles including poles at origin. For practical systems, n >m.

Since, $s \to R\,e^{j\theta}$ and $R \to 0$, the term 1+ sT can be approximated to 1, [i.e.$(1+sT) \approx 1$].

$$\therefore G(s)H(s) \approx K\frac{1}{s^y}$$

On letting, $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$ we get,

$$G(s)H(s)\Bigg|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to 0}{Lt}\ (Re^{j\theta})^y} = \infty e^{-j\theta y}$$

When $\theta = -\frac{\pi}{2}$, $G(s)H(s) = \infty e^{j\frac{\pi}{2}y}$

When $\theta = \frac{\pi}{2}$, $G(s)H(s) = \infty e^{-j\frac{\pi}{2}y}$

From the above two equations we can conclude that the section C_4 of Nyquist contour in s-plane is mapped as circles/circular arc in G(s)H(s)-plane with origin as centre and infinite radius.

Note :

1. *If there are no poles on the origin then the section C_4 of Nyquist contour will be absent.*
2. *If there are poles on imaginary axis as shown below then the Nyquist contour is divided into the following 8 sections and the mapping is performed sectionwise.*

Section C_1 : $s = j\omega$; $\omega = 0^+$ to $+\omega_1^-$

Section C_2 : $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$; $\theta = -\frac{\pi}{2}$ *to* $+\frac{\pi}{2}$

Section C_3 : $s = j\omega$; $\omega = +\omega_1^+$ *to* $+\infty$

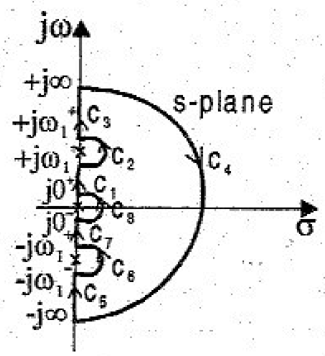
Section C_4 : $s = \underset{R\to \infty}{Lt}\ Re^{j\theta}$; $\theta = +\frac{\pi}{2}$ *to* $-\frac{\pi}{2}$

Section C_5 : $s = j\omega$; $\omega = -\infty$ *to* $-\omega_1^-$

Section C_6 : $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$; $\theta = -\frac{\pi}{2}$ *to* $+\frac{\pi}{2}$

Section C_7 : $s = j\omega$; $\omega = -\omega_1^+$ *to* 0^-

Section C_8 : $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$; $\theta = -\frac{\pi}{2}$ *to* $+\frac{\pi}{2}$



EXAMPLE 4.13

Draw the Nyquist plot for the system whose open loop transfer function is, $G(s)H(s) = \frac{K}{s\,(s+2)\,(s+10)}$.

Determine the range of K for which closed loop system is stable.

SOLUTION

$$\text{Given that, } G(s)H(s) = \frac{K}{s\,(s+2)\,(s+10)} = \frac{K}{s\times 2\left(\frac{s}{2}+1\right)\times 10\left(\frac{s}{10}+1\right)} = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 4.13.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

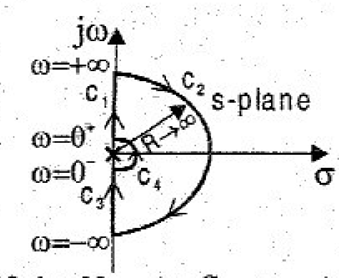


Fig 4.13.1 : *Nyquist Contour in s-plane*

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)}$$

Let $s = j\omega$.

$$\therefore G(j\omega)H(j\omega) = \frac{0.05K}{j\omega\,(1+j0.5\omega)\,(1+j0.1\omega)} = \frac{0.05K}{j\omega\,(1+j0.6\omega-0.05\omega^2)} = \frac{0.05K}{-0.6\omega^2 + j\omega\,(1-0.05\omega^2)}$$

When the locus of $G(j\omega)H(j\omega)$ crosses real axis the imaginary term will be zero and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}(1-0.05\omega_{pc}^2) = 0 \quad \Rightarrow \quad 1-0.05\omega_{pc}^2 = 0 \quad \Rightarrow \quad \omega_{pc} = \sqrt{\frac{1}{0.05}} = 4.472 \text{ rad / sec}$$

$$\text{At } \omega = \omega_{pc} = 4.472 \text{ rad / sec}, \qquad G(j\omega)H(j\omega) = \frac{0.05K}{-0.6\omega^2} = -\frac{0.05K}{0.6 \times (4.472)^2} = -0.00417K$$

The open loop system is type-1 and third order system. Also it is a minimum phase system with all poles. Hence the polar plot of $G(j\omega)H(j\omega)$ starts at $-90°$ axis at infinity, crosses real axis at $-0.00417K$ and ends at origin in second quadrant. The section C_1 and its mapping are shown in fig 4.13.2. and 4.13.3.

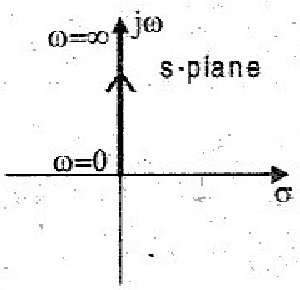


Fig 4.13.2 : Section C_1 in s-plane

⇒

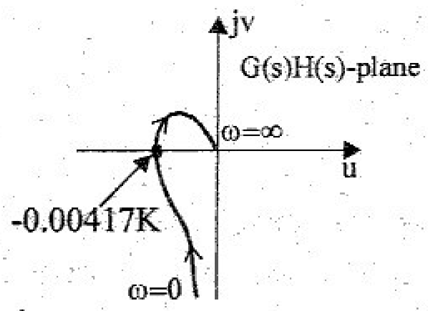


Fig 4.13.3 : *Mapping of section C_1 in G(s)H(s)-plane*

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\, R\,e^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s)H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)\,H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 0.5s \times 0.1s} = \frac{K}{s^3}$$

Let, $s = \underset{R\to\infty}{Lt}\; Re^{j\theta}$.

$$\therefore \; G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{s^3}\Bigg|_{s=\underset{R\to\infty}{Lt} Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt}\,(Re^{j\theta})^3} = 0e^{-j3\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \qquad G(s)H(s) = 0e^{-j3\frac{\pi}{2}} \qquad .....(1)$$

$$\text{When } \theta = -\frac{\pi}{2}, \qquad G(s)H(s) = 0e^{+j3\frac{\pi}{2}} \qquad .....(2)$$

From the equations (1) and (2) we can say that section C_2 in s-plane (fig 4.13.4.) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-3\pi/2$ to $+3\pi/2$ as shown in fig 4.13.5.

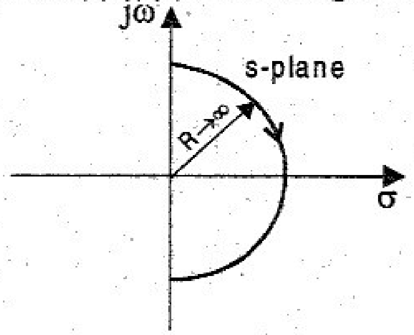


Fig 4.13.4 : Section C_2 in s-plane

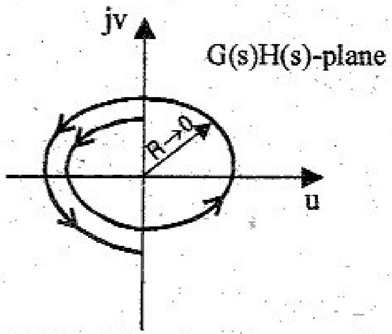


Fig 4.13.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of G(jω)H(jω) as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of G(jω)H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.13.6 and fig 4.13.7.

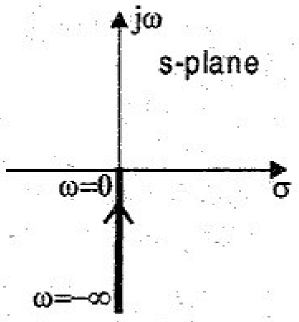


Fig 4.13.6 : Section C_3 in s-plane

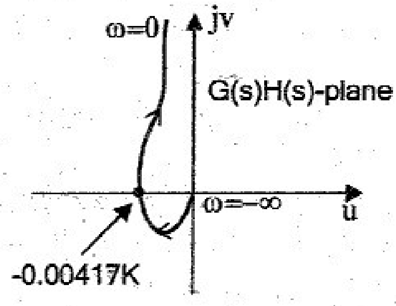


Fig 4.13.7 : *Mapping of section C_3 in G(s)H(s)-plane*

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to 0}{\text{Lt}}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s) H(s) can be approximated as shown below, [i.e., $(1+sT) \approx 1$].

$$G(s)H(s) = \frac{0.05K}{s\,(1+0.5s)\,(1+0.1s)} \approx \frac{0.05K}{s \times 1 \times 1} = \frac{0.05K}{s}$$

Let $s = \underset{R\to 0}{\text{Lt}}\ Re^{j\theta}$.

$$\therefore\ G(s)H(s)\Bigg|_{s=\underset{R\to 0}{\text{Lt}} Re^{j\theta}} = \frac{0.05K}{s}\Bigg|_{s=\underset{R\to 0}{\text{Lt}} Re^{j\theta}} = \frac{0.05K}{\underset{R\to 0}{\text{Lt}}\,(Re^{j\theta})} = \infty e^{-j\theta}$$

When $\theta = -\dfrac{\pi}{2}$, $\quad G(s)H(s) = \infty e^{+j\frac{\pi}{2}}$ .....(3)

When $\theta = \dfrac{\pi}{2}$, $\quad G(s)H(s) = \infty e^{-j\frac{\pi}{2}}$ .....(4)

From the equations (3) and (4) we can say that section C_4 in s-plane (fig 4.13.8.) is mapped as a circular arc of infinite radius with argument (phase) varying from $+\pi/2$ to $-\pi/2$ as shown in fig 4.13.9.

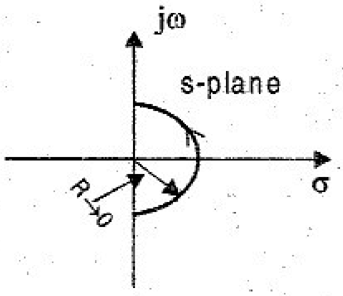


Fig 4.13.8 : Section C_4 in s-plane

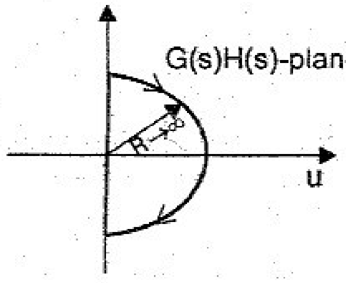


Fig 4.13.9 : *Mapping of section C_4 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.13.10.

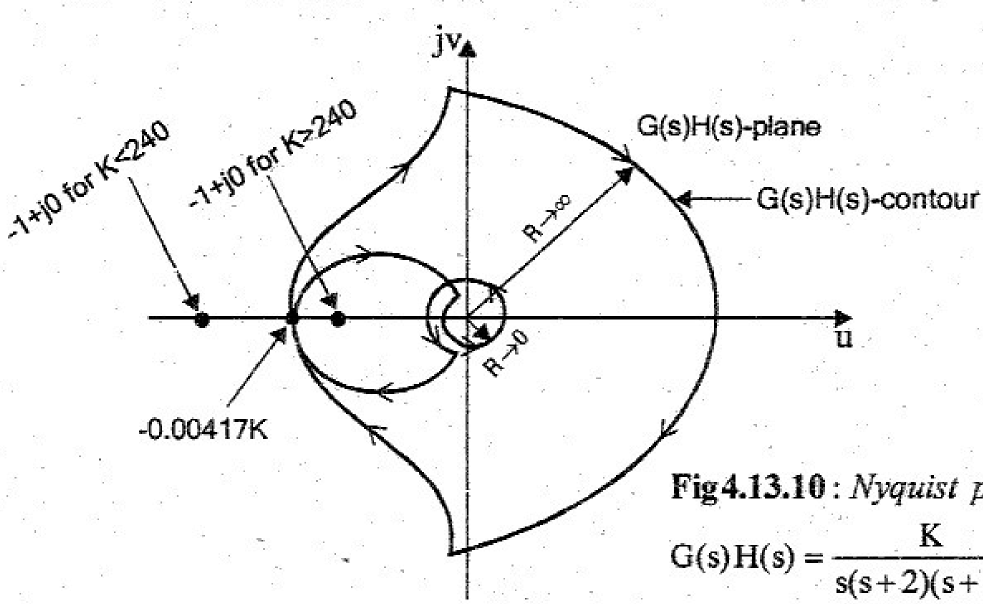


Fig 4.13.10 : *Nyquist plot of* $G(s)H(s) = \dfrac{K}{s(s+2)(s+10)}$

STABILITY ANALYSIS

When, $-0.00417K = -1$, the contour passes through $(-1+j0)$ point and corresponding value of K is the limiting value of K for stability.

$$\therefore \text{ Limiting value of } K = \frac{1}{0.00417} = 240$$

When K < 240

When K is less than 240, the contour crosses real axis at a point between 0 and $-1+j0$. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is not encircled. Also the open loop transfer function has no poles on the right half of s-plane. Therefore the closed loop system is stable.

When K > 240

When K is greater than 240, the contour crosses real axis at a point between $-1+j0$ and $-\infty$. On travelling through Nyquist plot along the indicated direction it is found that the point $-1+j0$ is encircled in clockwise direction two times. [Since there are two clockwise encirclement and no right half open loop poles, the closed loop system has two poles on right half of s-plane]. Therefore the closed loop system is unstable.

RESULT

The value of K for stability is $0 < K < 240$

EXAMPLE 4.14

Construct the Nyquist plot for a system whose open loop transfer function is given by $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$. Find the range of K for stability.

SOLUTION

Given that, $G(s)H(s) = \frac{K(1+s)^2}{s^3}$

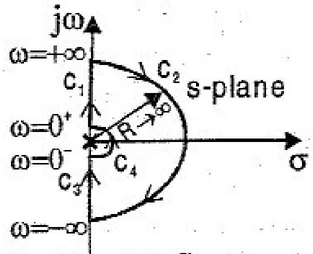


Fig 4.14.1 : *Nyquist Contour in s-plane*

The open loop transfer function has three poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 4.14.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to +∞. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{K(1+s)^2}{s^3}$$

Let $s = j\omega$.

$$\therefore G(j\omega)H(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3} = \frac{K(1-\omega^2+2j\omega)}{-j\omega^3} = \frac{K(1-\omega^2)}{-j\omega^3} + \frac{K\,2j\omega}{-j\omega^3} = -\frac{2K}{\omega^2} + j\frac{K(1-\omega^2)}{\omega^3}$$

When the $G(j\omega)H(j\omega)$ locus crosses real axis the imaginary term will be zero and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc}, \quad K(1-\omega_{pc}^2) = 0 \quad \Rightarrow \quad 1-\omega_{pc}^2 = 0 \quad \Rightarrow \quad \omega_{pc} = 1\text{ rad/sec}$$

At $\omega = \omega_{pc} = 1$ rad/sec,

$$G(j\omega)H(j\omega) = -\frac{2K}{\omega^2} = -\frac{2K}{1^2} = -2K \qquad .....(1)$$

$$G(j\omega)\,H(j\omega) = \frac{K(1+j\omega)^2}{(j\omega)^3} = \frac{K\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega}{\omega^3\angle 270°} = \frac{K(1+\omega^2)}{\omega^3}\angle(2\tan^{-1}\omega - 270°)$$

$$\text{As } \omega \to 0, \quad G(j\omega)H(j\omega) \to \infty\angle -270° \qquad .....(2)$$

$$\text{As } \omega \to \infty, \quad G(j\omega)H(j\omega) \to 0\angle -90° \qquad .....(3)$$

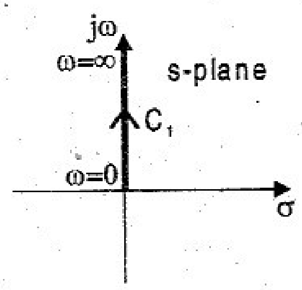


Fig 4.14.2 : *Section C_1 in s-plane*

⇒

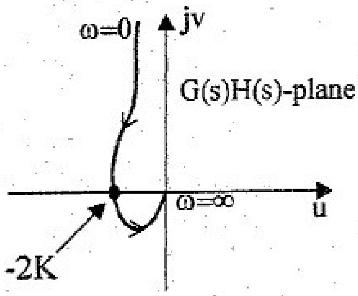


Fig 4.14.3 : *Mapping of section C_1 in G(s)H(s)-plane*

From equations (1), (2) and (3) we can say that the polar plot starts at −270° axis at infinity, crosses real axis at −2K and ends at origin in third quadrant. The section C_1 and its mapping are shown in fig 2 and 3.

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt} Re^{j\theta}$ in G(s)H(s) and varying θ from +π/2 to −π/2. Since $s \to Re^{j\theta}$ and $R \to \infty$, the G(s) H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)H(s) = \frac{K\,(1+s)^2}{s^3} \approx \frac{K\,s^2}{s^3} = \frac{K}{s}$$

Let $s = \underset{R\to\infty}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\ H(s)\Big|_{s=\underset{R\to\infty}{Lt}\ Re^{j\theta}} = \frac{K}{s}\Big|_{s=\underset{R\to\infty}{Lt}\ Re^{j\theta}} = \frac{K}{\underset{R\to\infty}{Lt}\ Re^{j\theta}} = 0e^{-j\theta}$$

When $\theta = \frac{\pi}{2}$, $\quad G(s)\ H(s) = 0e^{-j\frac{\pi}{2}}$ .....(4)

When $\theta = -\frac{\pi}{2}$, $\quad G(s)\ H(s) = 0e^{j\frac{\pi}{2}}$ .....(5)

From the equations (4) and (5) we can say that section C_2 in s-plane (fig 4.14.4.) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-\pi/2$ to $+\pi/2$ as shown in fig 4.14.5.

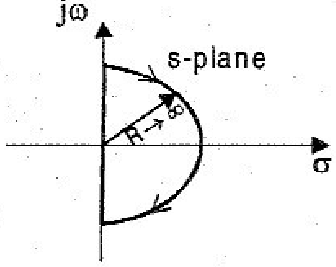


Fig 4.14.4 : *Section C_2 in s-plane*

⇒

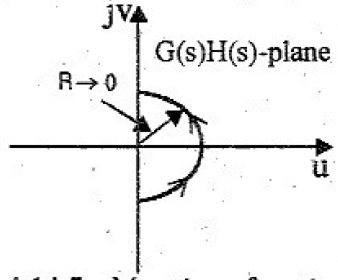


Fig 4.14.5 : *Mapping of section C_2 in G(s)H(s)-plane*

Mapping of section C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by locus of G(jω) H(jω) as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of G(jω) H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.14.6 and fig 4.14.7.

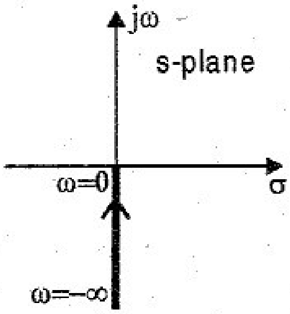


Fig 4.14.6 : *Section C_3 in s-plane*

⇒

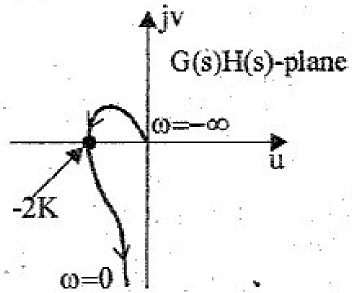


Fig 4.14.7 : *Mapping of section C_3 in G(s)H(s)-plane*

Mapping of section C_4

The mapping of section C_4 from s-plane to G(s) H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s)H(s) can be approximated as shown below, [i.e., $(1+sT) \approx 1$].

$$G(s)H(s) = \frac{K\,(1+s)^2}{s^3} \approx \frac{K \times 1}{s^3} = \frac{K}{s^3}$$

Let $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)\ H(s)\Big|_{s=\underset{R\to 0}{Lt}\ Re^{+j\theta}} = \frac{K}{s^3}\Big|_{s=\underset{R\to 0}{Lt}\ Re^{+j\theta}} = \frac{K}{\underset{R\to 0}{Lt}\,(Re^{+j\theta})^3} = \infty e^{-j3\theta}$$

When $\theta = -\frac{\pi}{2}$, $G(s)\,H(s) = \infty e^{+j3\frac{\pi}{2}}$ ......(6)

When $\theta = \frac{\pi}{2}$, $G(s)\,H(s) = \infty e^{-j3\frac{\pi}{2}}$ ......(7)

From the equations (6) and (7) we can say that section C_4 in s-plane (fig 4.14.8.) is mapped as a circular arc of infinite radius with argument (phase) varying from $+3\pi/2$ to $-3\pi/2$ as shown in fig 4.14.9.

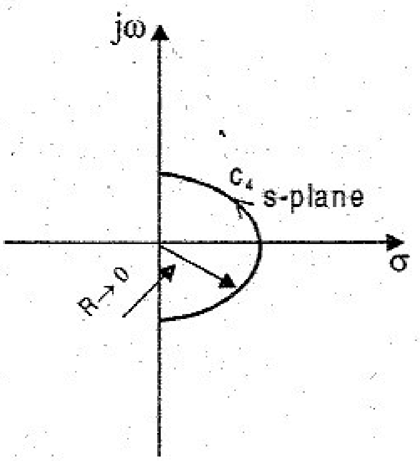


Fig 4.14.8 : *Section C_4 in s-plane*

$\Rightarrow$

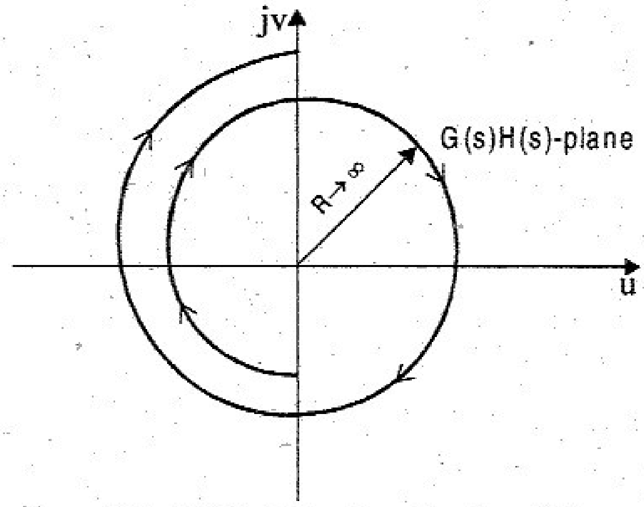


Fig 4.14.9 : *Mapping of section C_4 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.14.10.

STABILITY ANALYSIS

When, $-2K = -1$, the contour passes through $-1+j0$ point and corresponding value of K is the limiting value of K for stability.

$$\therefore \text{Limiting value of } K = \frac{1}{2} = 0.5$$

When K < 0.5

When K is less than 0.5, the contour crosses real axis at a point between 0 and $-1+j0$. On travelling through Nyquist plot along the indicated direction it is observed that the $-1+j0$ point is encircled in clockwise direction two times. Therefore the system is unstable. [Since there are two clockwise encirclement and no right half open loop poles, the closed loop system will have two poles on right half of s-plane]

When K > 0.5

When K is greater than 0.5, the contour crosses real axis at a point between $-1+j0$ and $-\infty$. On travelling through Nyquist plot along the indicated direction it is observed that $(-1+j0)$ point is encircled in both clockwise and anticlockwise direction one time. Hence net encirclement is zero. Also the open loop system has no poles at the right half of s-plane. Therefore the closed loop system is stable.

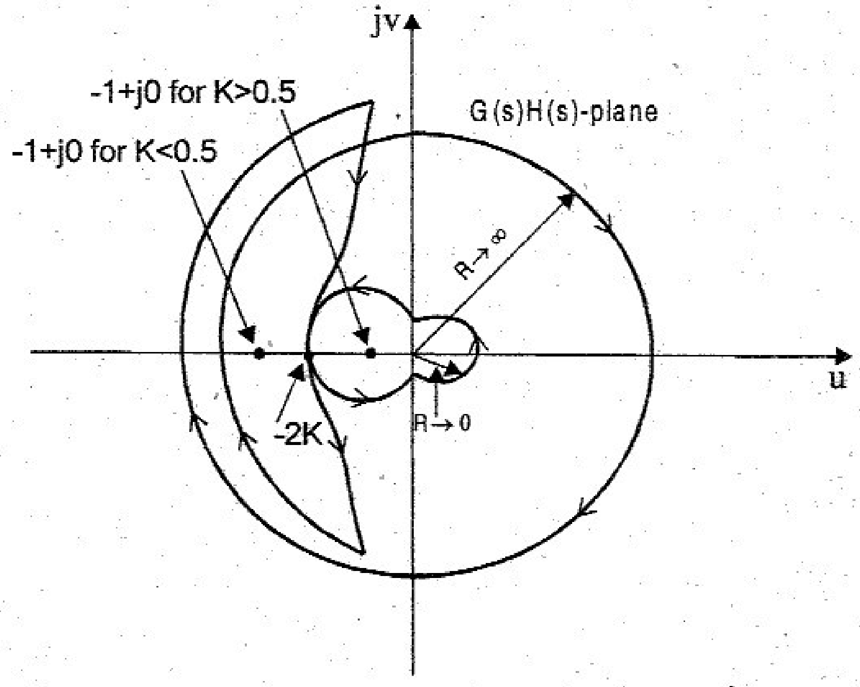


Fig 4.14.10 : Nyquist plot of $G(s)H(s) = \dfrac{K(1+s)^2}{s^3}$

RESULT

The system is stable when K > 0.5.

EXAMPLE 4.15

The open loop transfer function of a system is $G(s)H(s) = \dfrac{(1+4s)}{s^2(1+s)\,(1+2s)}$. Determine the stability of closed loop system. If the closed loop system is not stable then find the number of closed-loop poles lying on the right half of s-plane.

SOLUTION

Given that, $G(s)H(s) = \dfrac{(1+4s)}{s^2(1+s)\,(1+2s)}$

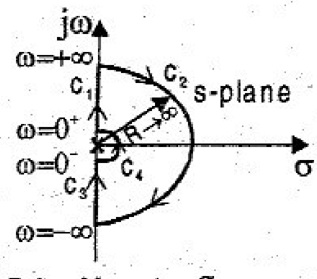


Fig 4.15.1 : *Nyquist Contour in s-plane*

The open loop transfer function has two poles at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane except the origin as shown in fig 4.15.1.

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)}$$

Let $s = j\omega$.

$$\therefore G(j\omega)H(j\omega) = \frac{(1+j4\omega)}{(j\omega)^2\,(1+j\omega)\,(1+j2\omega)} = \frac{\sqrt{1+16\omega^2}\,\angle\tan^{-1}4\omega}{\omega^2\,\angle 180^\circ\sqrt{1+\omega^2}\,\angle\tan^{-1}\omega\,\sqrt{1+4\omega^2}\,\angle\tan^{-1}2\omega}$$

$$= \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\,\sqrt{1+4\omega^2}}\,\angle(\tan^{-1}4\omega - 180^\circ - \tan^{-1}\omega - \tan^{-1}2\omega)$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{\sqrt{1+16\omega^2}}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}4\omega - 180° - \tan^{-1}\omega - \tan^{-1}2\omega$$

When the $G(j\omega)H(j\omega)$ locus crosses real axis, the phase will be $-180°$ and the corresponding frequency is the phase crossover frequency, ω_{pc}.

$$\therefore \text{At } \omega = \omega_{pc},\ \angle G(j\omega)H(j\omega) = -180°$$

$$\therefore \tan^{-1}4\omega_{pc} - 180° - \tan^{-1}\omega_{pc} - \tan^{-1}2\omega_{pc} = -180°$$

$$\tan^{-1}4\omega_{pc} = \tan^{-1}\omega_{pc} + \tan^{-1}2\omega_{pc}$$

On taking tan on both sides we get,

$$\tan\ [\tan^{-1}4\omega_{pc}] = \tan\ [\tan^{-1}\omega_{pc} + \tan^{-1}2\omega_{pc}]$$

$$4\omega_{pc} = \frac{\tan\ \tan^{-1}\omega_{pc} + \tan\ \tan^{-1}2\omega_{pc}}{1 - \tan\ \tan^{-1}\omega_{pc}\ \times\ \tan\ \tan^{-1}2\omega_{pc}}$$

Note: $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \times \tan B}$

$$4\omega_{pc} = \frac{\omega_{pc} + 2\omega_{pc}}{1 - 2\omega_{pc}^2} \quad \Rightarrow \quad 1 - 2\omega_{pc}^2 = \frac{3\omega_{pc}}{4\omega_{pc}} \quad \Rightarrow \quad -2\omega_{pc}^2 = \frac{3}{4} - 1$$

$$\therefore\ \omega_{pc} = \sqrt{\frac{-0.25}{-2}} = 0.354 \text{ rad / sec}$$

At $\omega = \omega_{pc} = 0.354$ rad / sec,

$$|G(j\omega)H(j\omega)| = \frac{\sqrt{1+16\omega_{pc}^2}}{\omega^2\sqrt{1+\omega_{pc}^2}\ \sqrt{1+4\omega_{pc}^2}} = \frac{\sqrt{1+16\times 0.354^2}}{(0.354)^2\ \sqrt{1+0.354^2}\ \sqrt{1+4\times 0.354^2}} = 10.64 \quad .....(1)$$

Hence $G(j\omega)H(j\omega)$ locus crosses the real axis at -10.64.

$$\text{At } \omega \to 0,\ G(j\omega)H(j\omega) \to \infty\angle -180° \quad .....(2)$$

$$\text{At } \omega \to \infty,\ G(j\omega)H(j\omega) \to 0\angle -270° \quad .....(3)$$

From equations (1), (2) and (3) we can say that the polar plot starts at $-180°$ axis at infinity, travels in third quadrant and crosses real axis at -10.64 to enter second quadrant and then ends at origin in second quadrant. The section C_1 and its mapping are shown in fig 4.15.2. and 4.15.3.

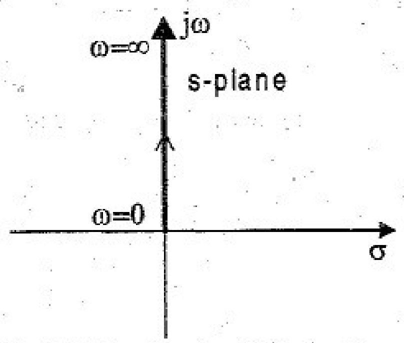


Fig 4.15.2 : *Section C_1 in s-plane*

⇒

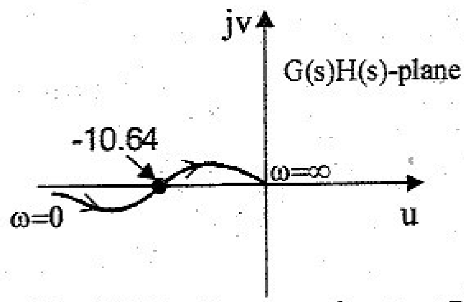


Fig 4.15.3 : *Mapping of section C_1 in G(s)H(s)-plane*

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s)H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)} \approx \frac{4s}{s^2 \times s \times 2s} = \frac{2}{s^3}$$

Let, $s = \underset{R\to\infty}{\text{Lt}}\ Re^{j\theta}$.

$$\therefore\ G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{\text{Lt}}\ Re^{j\theta}} = \frac{2}{s^3}\Bigg|_{s=\underset{R\to\infty}{\text{Lt}}\ Re^{j\theta}} = \frac{2}{\underset{R\to\infty}{\text{Lt}}\ (Re^{j\theta})^3} = 0e^{-j3\theta}$$

When $\theta = \frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{-j3\frac{\pi}{2}}$ .....(4)

When $\theta = -\frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{j3\frac{\pi}{2}}$ .....(5)

From the equations (4) and (5) we can say that section C_2 in s-plane (fig 4.15.4.) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument (phase) varying from $-3\pi/2$ to $+3\pi/2$ as shown in fig 4.15.5.

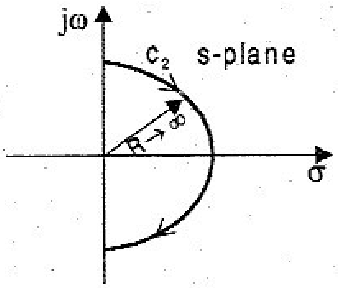


Fig 4.15.4 : *Section C_2 in s-plane*

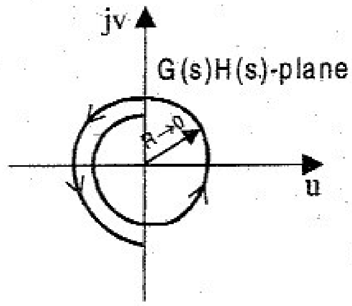


Fig 4.15.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)\,H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.15.6 and fig 4.15.7.

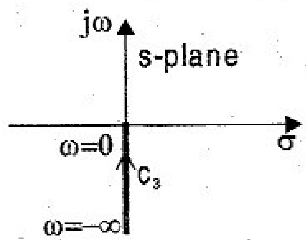


Fig 4.15.6 : *Section C_3 in s-plane*

$\Rightarrow$

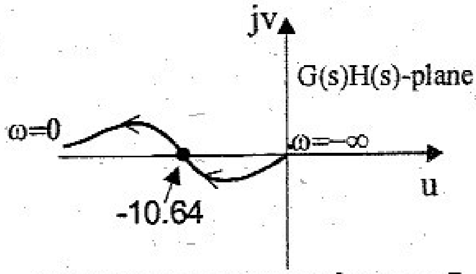


Fig 4.15.7 : *Mapping of section C_3 in G(s)H(s)-plane*

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to 0}{\text{Lt}}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, the G(s)H(s) can be approximated as shown below [i.e., $(1+sT) \approx 1$].

$$G(s)H(s) = \frac{(1+4s)}{s^2(1+s)\,(1+2s)} \approx \frac{1}{s^2 \times 1 \times 1} = \frac{1}{s^2}$$

Let, $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$

$$\therefore\ G(s)H(s)\Bigg|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{1}{s^2}\Bigg|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{1}{\underset{R\to 0}{Lt}(Re^{j\theta})^2} = \infty e^{-j2\theta}$$

When $\theta = -\frac{\pi}{2}$, $G(s)H(s) = \infty e^{j\pi}$ .....(6)

When $\theta = \frac{\pi}{2}$, $G(s)H(s) = \infty e^{-j\pi}$ .....(7)

From the equations (6) and (7) we can say that section C_4 in s-plane (fig 4.15.8.) is mapped as a circle of infinite radius with argument (phase) varying from $+\pi$ to $-\pi$ as shown in fig 4.15.9.

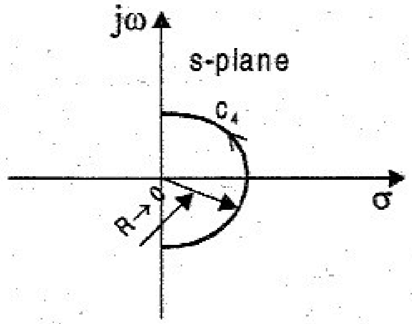


Fig 4.15.8 : *Section C_4 in s-plane*

⇒

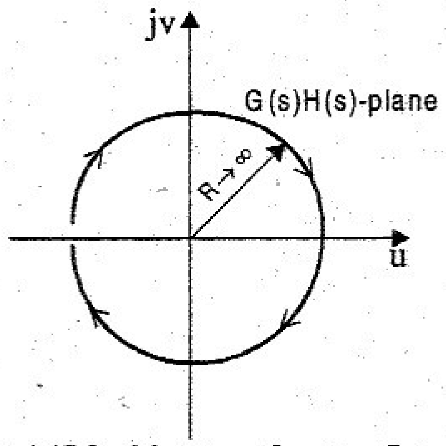


Fig 4.15.9 : *Mapping of section C_4 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.15.10.

STABILITY ANALYSIS

On travelling through Nyquist contour in G(s)H(s)-plane it is observed that (–1+j0) point is encircled in clockwise direction two times. Therefore the closed loop system is unstable.

Since the –1+j0 is encircled two times in clockwise and no right half open loop poles, two poles of closed loop system are lying on the right half s-plane.

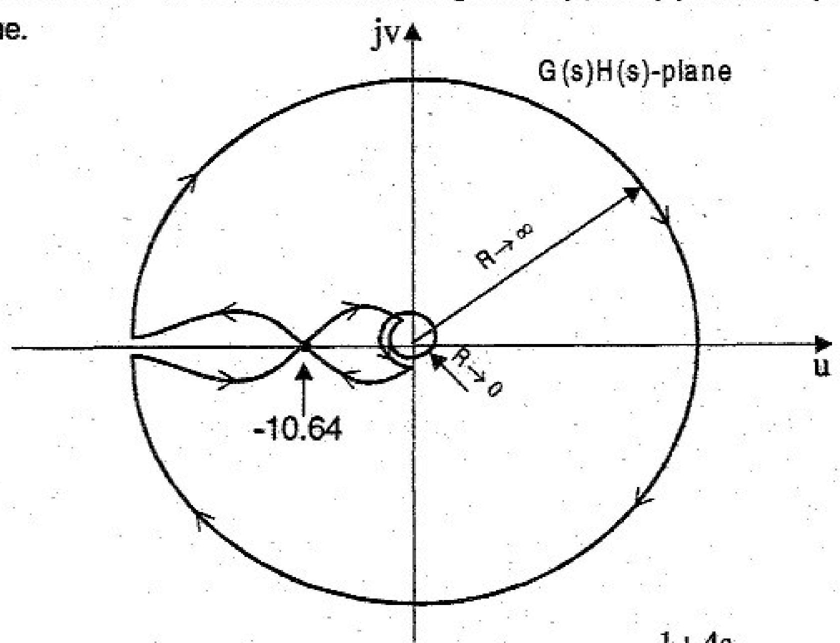


Fig 4.15.10 : Nyquist plot of $G(s)\,H(s) = \dfrac{1+4s}{s^2(1+s)(1+2s)}$

RESULT

(a) Closed loop system is unstable.

(b) Two poles of closed loop system are lying on the right half s-plane.

EXAMPLE 4.16

Sketch the Nyquist plot for a system with the open loop transfer function $G(s)H(s)=\dfrac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$. Determine the range of values of K for which the system is stable.

SOLUTION

Given that, $G(s)H(s)=\dfrac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane as shown in fig 4.16.1.

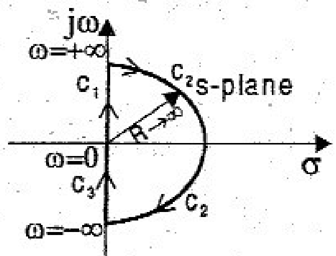


Fig 4.16.1 : *Nyquist Contour in s-plane*

The Nyquist contour has three sections C_1, C_2 and C_3. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s)=\frac{K\,(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$$

Let $s=j\omega$.

$$\therefore G(j\omega)H(j\omega)=\frac{K\,(1+j0.5\omega)\,(1+j\omega)}{(1+j10\omega)\,(-1+j\omega)}=\frac{K(1+j1.5\omega-0.5\omega^2)}{-1-j9\omega-10\omega^2}=\frac{K(1-0.5\omega^2)+j1.5\omega K}{-(1+10\omega^2)-j9\omega}$$

On multiplying the numerator and denominator by the complex conjugate of denominator we get,

$$G(j\omega)H(j\omega)=\frac{K(1-0.5\omega^2)+j1.5\omega K}{-(1+10\omega^2)-j9\omega}\times\frac{-(1+10\omega^2)+j9\omega}{-(1+10\omega^2)+j9\omega}$$

$$=\frac{-K(1-0.5\omega^2)\,(1+10\omega^2)-13.5\omega^2K+j\,[9\omega K(1-0.5\omega^2)-1.5\omega K(1+10\omega^2)]}{(1+10\omega^2)^2+(9\omega)^2}$$

When the $G(j\omega)H(j\omega)$ locus crosses real axis the imaginary term is zero and the corresponding frequency is the phase crossover frequency.

$$\therefore \text{At } \omega=\omega_{pc},\quad 9\omega_{pc}K(1-0.5\omega_{pc}^2)-1.5\omega_{pc}K(1+10\omega_{pc}^2)=0$$

$$\therefore 9\omega_{pc}K(1-0.5\omega_{pc}^2)=1.5\omega_{pc}K(1+10\omega_{pc}^2)\quad\Rightarrow\quad 1-0.5\omega_{pc}^2=\frac{1.5}{9}(1+10\omega_{pc}^2)$$

$$\therefore 1-0.5\omega_{pc}^2=0.167+1.67\omega_{pc}^2\quad\Rightarrow\quad 2.17\omega_{pc}^2=0.833\quad\Rightarrow\quad \omega_{pc}=\sqrt{\frac{0.833}{2.17}}=0.62\text{ rad/sec}$$

At $\omega=\omega_{pc}=0.62$ rad/sec

$$G(j\omega)H(j\omega) = \frac{-K(1-0.5\omega_{pc}^{2})\,(1+10\omega_{pc}^{2}) - 13.5\omega_{pc}^{2}K}{(1+10\omega_{pc}^{2})^2 + (9\omega_{pc})^2}$$

$$= -K\left[\frac{(1-0.5\times 0.62^2)\,(1+10\times 0.62^2) + 13.5\times 0.62^2}{(1+10\times 0.62^2)^2 + (9\times 0.62)^2}\right] = -K\left[\frac{3.913+5.189}{23.464+31.136}\right] = -0.1667K$$

Therefore, $G(j\omega)H(j\omega)$ locus crosses real axis at a point -0.1667K.

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude and phase of $G(j\omega)H(j\omega)$ for various values of ω.

$$G(j\omega)H(j\omega) = K\frac{(1+j0.5\omega)\,(1+j\omega)}{(1+j10\omega)\,(-1+j\omega)} = K\frac{\sqrt{1+(0.5\omega)^2}\ \angle\tan^{-1}0.5\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega}{\sqrt{1+(10\omega)^2}\ \angle\tan^{-1}10\omega\ \sqrt{1+\omega^2}\ \angle(180°-\tan^{-1}\omega)}$$

$$= K\frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}\ \angle(\tan^{-1}0.5\omega + 2\tan^{-1}\omega - \tan^{-1}10\omega - 180°)$$

$$\therefore\ |G(j\omega)H(j\omega)| = K\frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = \tan^{-1}0.5\omega + 2\tan^{-1}\omega - \tan^{-1}10\omega - 180°$$

As $\omega \to 0,\ |G(j\omega)H(j\omega)| = K$

As $\omega \to 0,\ \angle G(j\omega)H(j\omega) = -180°$

$$\text{As } \omega\to\infty,\ |G(j\omega)H(j\omega)| = \underset{\omega\to\infty}{Lt} K\frac{\sqrt{1+0.25\omega^2}}{\sqrt{1+100\omega^2}} = K\underset{\omega\to\infty}{Lt}\sqrt{\frac{\omega^2\left(\frac{1}{\omega^2}+0.25\right)}{\omega^2\left(\frac{1}{\omega^2}+100\right)}} = K\underset{\omega\to\infty}{Lt}\sqrt{\frac{\left(\frac{1}{\omega^2}+0.25\right)}{\left(\frac{1}{\omega^2}+100\right)}} = K\sqrt{\frac{0+0.25}{0+100}} = 0.05K$$

As $\omega \to \infty,\ \angle G(j\omega) = \tan^{-1}\infty + 2\tan^{-1}\infty - \tan^{-1}\infty - 180° = 90° + 180° - 90° - 180° = 0$

ω rad/sec	0	0.1	0.5	1.5	2.0	5.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	K	0.707K	0.202K	0.083K	0.07K	0.054K	0.05K
$\angle G(j\omega)H(j\omega)$ deg	−180	−210	−191	−116	−95	−43	0

From the above analysis, the following conclusions are made,

1. The locus of $G(j\omega)H(j\omega)$ starts at $K\angle -180°$ when $\omega = 0$ and travels in second quadrant.
2. The locus crosses real axis at -0.1667K and enters third quadrant.
3. Then the locus crosses negative imaginary axis and enters fourth quadrant.
4. Finally the locus ends at $0.05K\angle 0^0$ when $\omega = \infty$.

***Note:** The exact plot can also be sketched on polar graph sheet.*

The section C_1 in s-plane and its corresponding mapping in G(s)H(s) plane are shown in fig 4.16.2. and 4.16.3.

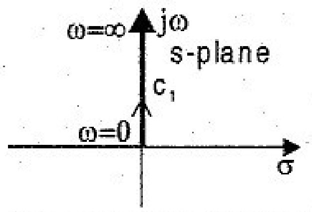


Fig 4.16.2 : *Section C_1 in s-plane*

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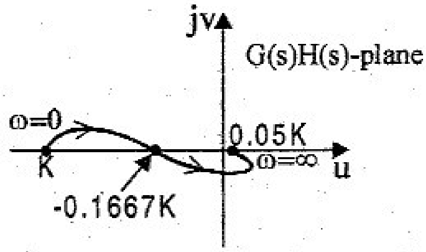


Fig 4.16.3 : *Mapping of section C_1 in G(s)H(s)-plane*

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt}\ R e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R e^{j\theta}$ and $R\to\infty$, G(s)H(s) can be approximated as shown below [i.e., $(1+sT) \approx sT$; Here $(s-1) \approx s$].

$$G(s)H(s) = \frac{K(1+0.5s)\,(1+s)}{(1+10s)\,(s-1)}$$

$$\approx \frac{K\,0.5s \times s}{10s\ \times\ s} = 0.05K$$

The approximate G(s)H(s) is independent of s and so the contour of section C_2 in s-plane is mapped as a point at 0.05K in G(s)H(s)-plane.

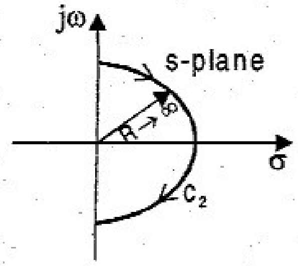


Fig 4.16.4 : *Section C_2 in s-plane*

⇒

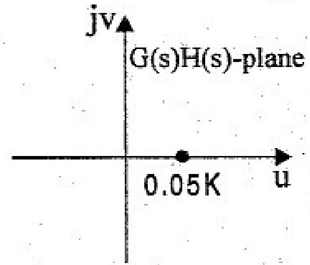


Fig 4.16.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)\,H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)\,H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.16.6 and fig 4.16.7.

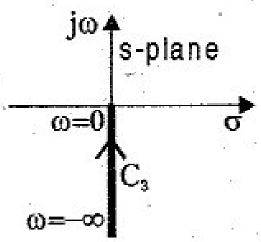


Fig 4.16.6 : *Section C_3 in s-plane*

⇒

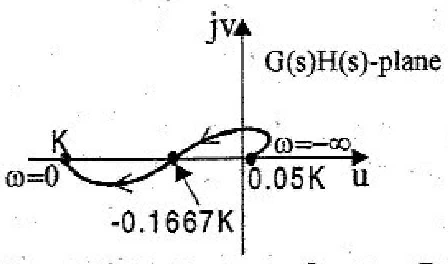


Fig 4.16.7 : *Mapping of section C_3 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.16.8.

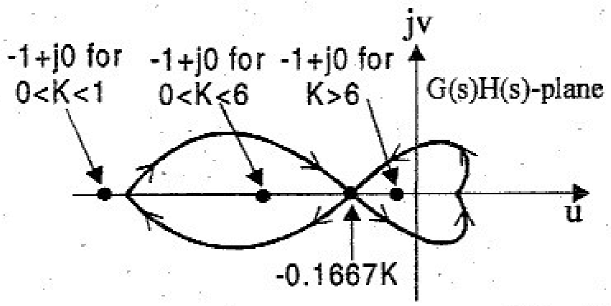


Fig 4.16.8 : *Nyquist plot of* $G(s)\,H(s) = \dfrac{K(1+0.5s)(1+s)}{(1+10s)(s-1)}$

STABILITY ANALYSIS

When $-0.1667K = -1$, the contour passes through $-1+j0$ point and this value of K is the limiting value for stability.

The limiting value of $K = \frac{1}{0.1667} = 6$

When 0<K<1

When 0<K<1, the $-1+j0$ point is not encircled, but there is one open loop right half pole and so system is unstable.

When 1<K<6

When 1<K<6, the locus crosses real axis between 0 and $-1+j0$. On travelling through the locus it is observed that the $-1+j0$ point is encircled clockwise and so the closed loop system is unstable.

When K>6

When K>6, the locus crosses real axis between $-1+j0$ and $-\infty$. On travelling through the locus it is observed that the $-1+j0$ point is encircled anticlockwise one time. Also the open loop system has one pole at the right half s-plane. Hence the system is stable.

RESULT

(a) The open loop system is unstable.

(b) For stability of the closed loop system, K>6.

EXAMPLE 4.17

Construct Nyquist plot for a feedback control system whose open loop transfer function is given by, $G(s)H(s) = \frac{5}{s(1-s)}$ Comment on the stability of open-loop and closed loop system.

SOLUTION

Given that, $G(s)H(s) = \frac{5}{s(1-s)}$

The open loop transfer function has a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half s-plane except the origin as shown in fig 4.17.1.

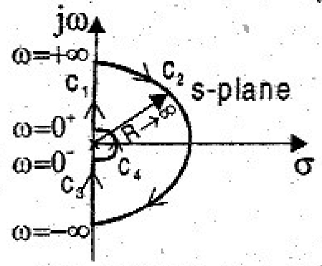


Fig 4.17.1 : *Nyquist Contour in s-plane*

The Nyquist contour has four sections C_1, C_2, C_3 and C_4. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

$$G(s)H(s) = \frac{5}{s(1-s)}$$

Let $s = j\omega$. $\therefore G(j\omega)H(j\omega) = \frac{5}{j\omega(1-j\omega)}$

$$= \frac{5}{\omega\angle 90^\circ \sqrt{1+\omega^2}\angle -\tan^{-1}\omega}$$

$$= \frac{5}{\omega\sqrt{1+\omega^2}}\angle(-90+\tan^{-1}\omega)$$

$$\therefore |G(j\omega)H(j\omega)| = \frac{5}{\omega\sqrt{1+\omega^2}}$$

$$\angle G(j\omega)H(j\omega) = -90^\circ + \tan^{-1}\omega$$

Note: $(1-j\omega)$ represents a point in fourth quadrant

The exact shape of $G(j\omega)H(j\omega)$ locus is determined by calculating the magnitude and phase of $G(j\omega)H(j\omega)$ for various values of ω.

ω rad/sec	0	0.6	1.0	2.0	10.0	∞
$\lvert G(j\omega)H(j\omega)\rvert$	∞	7.15	3.53	1.12	0.05	0
$\angle G(j\omega)H(j\omega)$ deg	−90	−59	−45	−26	−5	0

From the above analysis, we can conclude that $G(j\omega)H(j\omega)$ locus starts at $-90°$ axis at infinity for $\omega = 0$ and meets the origin along $0°$ axis when $\omega = \infty$.

The section C_1 in s-plane and its corresponding mapping in G(s)H(s)-plane are shown in fig 4.17.2 and 4.17.3.

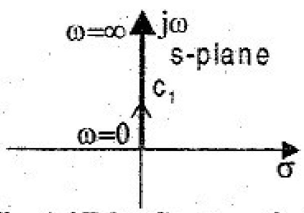


Fig 4.17.2 : Section C_1 in s-plane

$\Rightarrow$

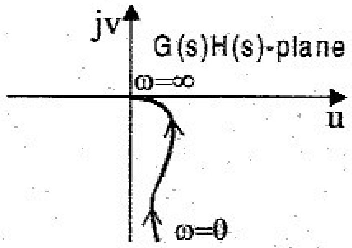


Fig 4.17.3 : Mapping of section C_1 in G(s)H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{\mathrm{Lt}}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to \infty$, the G(s)H(s) can be approximated as shown below, [i.e., $(1-s) \approx -s$)]

$$G(s)H(s) = \frac{5}{s(1-s)} \approx \frac{5}{s\,(-s)} = \frac{5}{s^2\,e^{j\pi}}$$

Let, $s = \underset{R\to\infty}{\mathrm{Lt}}\ R\,e^{j\theta}$.

Note : $-1 = e^{j\pi}$

$$\therefore G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{\mathrm{Lt}}\ R\,e^{j\theta}} = \frac{5}{\underset{R\to\infty}{\mathrm{Lt}}\ (R\,e^{j\theta})^2 e^{j\pi}} = 0e^{-j(2\theta+\pi)}$$

When $\theta = \frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{-j2\pi}$ .....(1)

When $\theta = -\frac{\pi}{2}$, $\quad G(s)H(s) = 0e^{j0}$ .....(2)

From the equations (1) and (2) we can say that section C_2 in s-plane (fig 4.17.4) is mapped as circular arc of zero radius around origin in G(s)H(s) plane with argument varying from -2π to $+0$ as shown in fig 4.17.5.

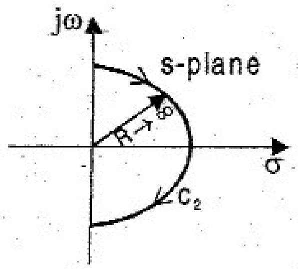


Fig 4.17.4 : Section C_2 in s-plane

$\Rightarrow$

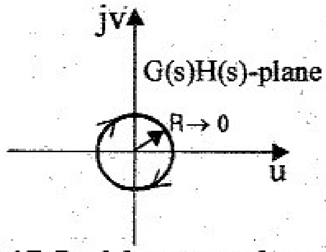


Fig 4.17.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.17.6 and fig 4.17.7.

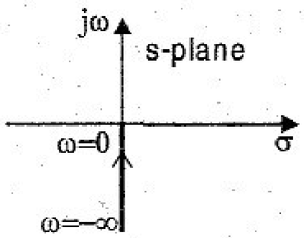


Fig 4.17.6 : Section C_3 in s-plane

⇒

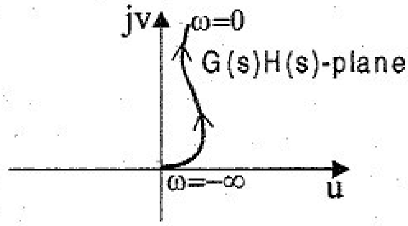


Fig 4.17.7 : Mapping of section C_3 in G(s)H(s)-plane

MAPPING OF SECTION C_4

The mapping of section C_4 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to 0}{Lt}\ R\,e^{j\theta}$ in G(s)H(s) and varying θ from $-\pi/2$ to $+\pi/2$. Since $s \to R\,e^{j\theta}$ and $R \to 0$, G(s)H(s) can be approximated as shown below, [i.e., $(1-s) \approx 1$].

$$G(s)H(s) = \frac{5}{s\,(1-s)} \approx \frac{5}{s \times 1} = \frac{5}{s}$$

Let $s = \underset{R\to 0}{Lt}\ Re^{j\theta}$.

$$\therefore\ G(s)H(s)\Bigg|_{s=\underset{R\to 0}{Lt} Re^{j\theta}} = \frac{5}{\underset{R\to 0}{Lt}\ Re^{j\theta}} = \infty e^{-j\theta}$$

$$\text{When } \theta = -\frac{\pi}{2},\quad G(s)H(s) = \infty e^{j\frac{\pi}{2}} \qquad .....(3)$$

$$\text{When } \theta = \frac{\pi}{2},\quad G(s)H(s) = \infty e^{-j\frac{\pi}{2}} \qquad .....(4)$$

From the equations (3) and (4) we can say that section C_4 in s-plane (fig 4.17.8.) is mapped as a circular arc of infinite radius with argument varying from $\pi/2$ to $-\pi/2$ as shown in fig 4.17.9.

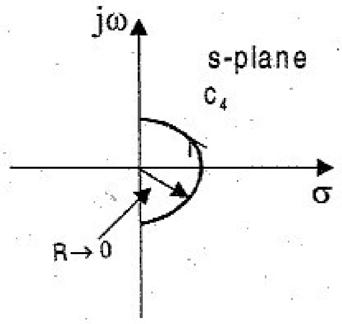


Fig 4.17.8 : Section C_4 in s-plane

⇒

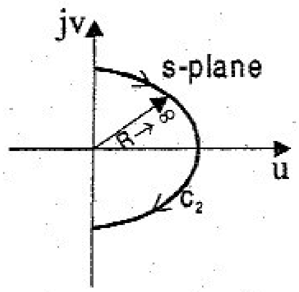


Fig 4.17.9 : Mapping of section C_4 in G(s)H(s)-plane

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.17.10.

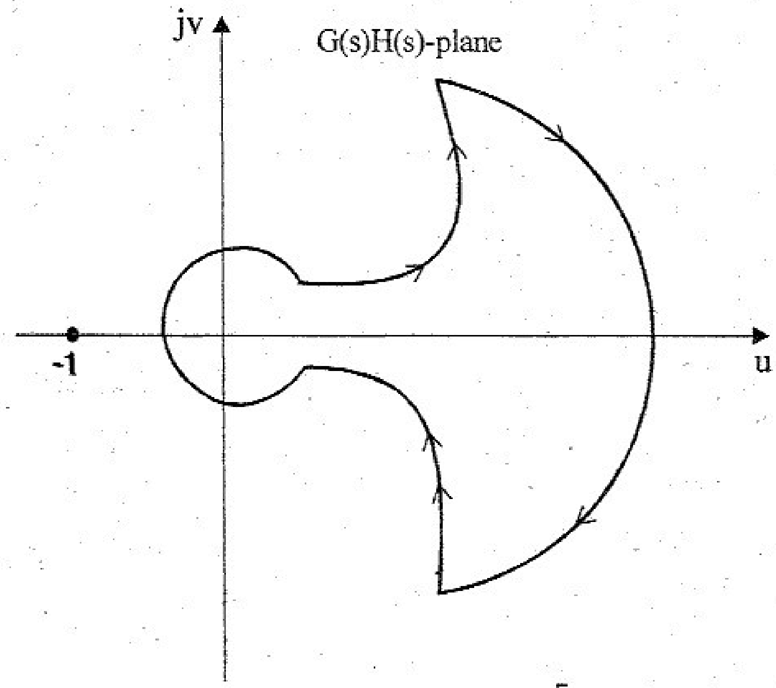


Fig 4.17.10 : Nyquist plot of $G(s)H(s) = \dfrac{5}{s(1-s)}$

STABILITY ANALYSIS

The Nyquist contour in G(s)H(s)-plane does not encircle the point $(-1+j0)$ but the open loop transfer function has one pole on the right half s-plane. Therefore the system is unstable.

RESULT

Both open loop and closed loop systems are unstable.

EXAMPLE 4.18

By Nyquist stability criterion determine the stability of closed loop system, whose open loop transfer function is given by,

$G(s)H(s) = \dfrac{(s+2)}{(s+1)\,(s-1)}$. Comment on the stability of open-loop and closed loop system.

SOLUTION

Given that, $G(s)H(s) = \dfrac{(s+2)}{(s+1)\,(s-1)}$

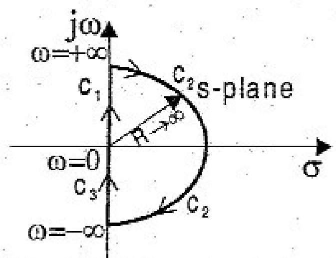


Fig 4.18.1 : *Nyquist Contour in s-plane*

The open loop transfer function does not have a pole at origin. Hence choose the Nyquist contour on s-plane enclosing the entire right half plane as shown in fig 4.18.1.

The Nyquist contour has three sections C_1, C_2 and C_3. The mapping of each section is performed separately and the overall Nyquist plot is obtained by combining the individual sections.

MAPPING OF SECTION C_1

In section C_1, ω varies from 0 to $+\infty$. The mapping of section C_1 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from 0 to ∞. This locus is the polar plot of $G(j\omega)H(j\omega)$.

> ***Note :** $(-1+j\omega)$ represents a point in second quadrant*

$$G(s)H(s) = \frac{s+2}{(s+1)\,(s-1)} = \frac{2(1+0.5s)}{(1+s)\,(-1+s)}$$

Let $s = j\omega$. $\quad \therefore G(j\omega)H(j\omega) = \dfrac{2(1+j0.5\omega)}{(1+j\omega)\,(-1+j\omega)} = \dfrac{2\sqrt{1+0.25\omega^2}\;\angle\tan^{-1}0.5\omega}{\sqrt{1+\omega^2}\;\angle\tan^{-1}\omega\;\sqrt{1+\omega^2}\;\angle(180^\circ-\tan^{-1}\omega)}$

$$= \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2} \angle(-180+\tan^{-1}0.5\omega)$$

$$\therefore \; |G(j\omega)H(j\omega)| = \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}$$

$$\angle G(j\omega)H(j\omega) = -180° + \tan^{-1}0.5\omega$$

The exact shape of G(jω)H(jω) locus is determined by calculating the magnitude and phase of G(jω)H(jω) for various values of ω.

ω rad/sec	0	0.4	1.0	2.0	10.0	∞
|G(jω)H(jω)|	2	1.76	1.12	0.57	0.1	0
∠G(jω)H(jω) deg	−180	−168	−153	−135	−101	−90

From the above analysis, we can conclude that G(jω)H(jω) locus starts at −180° axis at a magnitude of −2 for ω = 0 and meets the origin along −90° axis when ω = +∞.

The section C_1 in s-plane and its corresponding mapping in G(s)H(s)-plane are shown in fig 4.18.2. and 4.18.3.

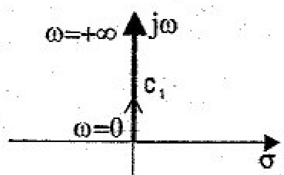


Fig 4.18.2 : Section C_1 in s-plane

⇒

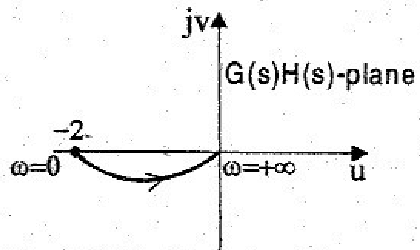


Fig 4.18.3 : Mapping of section C_1 in G(s)H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{Lt} \; R e^{j\theta}$ in G(s)H(s) and varying θ from $+\pi/2$ to $-\pi/2$. Since $s \to R e^{j\theta}$ and $R \to \infty$, G(s) H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)H(s) = \frac{2(1+0.5s)}{(1+s)\,(-1+s)} \approx \frac{2\times 0.5s}{s\times s} = \frac{1}{s}$$

Let, $s = \underset{R\to\infty}{Lt} \; R e^{j\theta}$.

$$\therefore G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{Lt} R e^{j\theta}} = \frac{1}{\underset{R\to\infty}{Lt} \; R e^{j\theta}} = 0e^{-j\theta}$$

$$\text{When } \theta = \frac{\pi}{2}, \quad G(s)H(s) = 0e^{-j\frac{\pi}{2}} \qquad .....(1)$$

$$\text{When } \theta = -\frac{\pi}{2}, \quad G(s)H(s) = 0e^{j\frac{\pi}{2}} \qquad .....(2)$$

From the equations (1) and (2) we can say that section C_2 in s-plane (fig 4.18.4) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument varying from $-\pi/2$ to $+\pi/2$ as shown in fig 4.18.5.

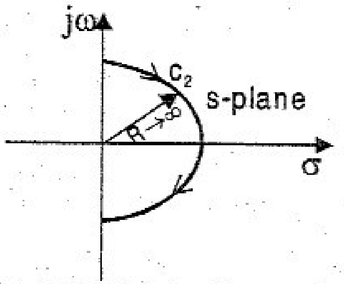


Fig 4.18.4 : *Section C_1 in s-plane*

$\Rightarrow$

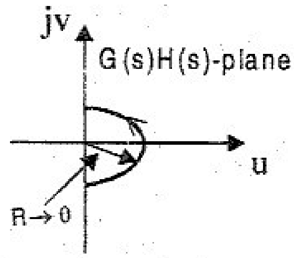


Fig 4.18.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from $-\infty$ to 0. The mapping of section C_3 is given by the locus of $G(j\omega)H(j\omega)$ as ω is varied from $-\infty$ to 0. This locus is the inverse polar plot of $G(j\omega)H(j\omega)$.

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.18.6 and fig 4.18.7.

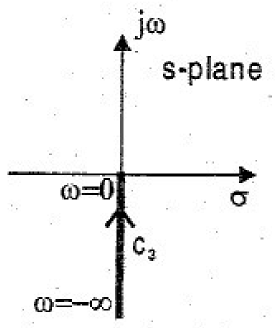


Fig 4.18.6 : *Section C_3 in s-plane*

$\Rightarrow$

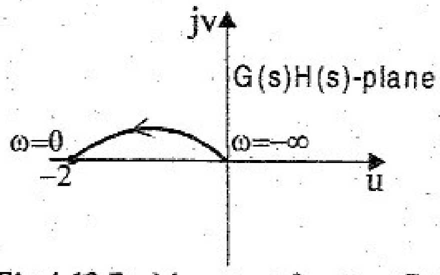


Fig 4.18.7 : *Mapping of section C_3 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.18.8.

STABILITY ANALYSIS

On travelling through Nyquist contour it is observed that $-1+j0$ point is encircled in anticlockwise direction one time. Also the open loop transfer function has one pole at right half s-plane. Since the number of anticlockwise encirclement is equal to number of open loop poles on right half s-plane, the closed loop system is stable.

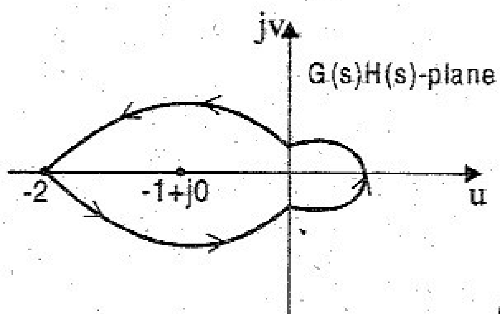


Fig 4.18.8 : Nyquist plot of $G(s)\,H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$

RESULT

(a) Open loop system is unstable

(b) Closed loop system is stable.

4.6 RELATIVE STABILITY

The ***Relative stability*** indicates the closeness of the system to stable region. It is an indication of the strength or degree of stability.

In time domain, the relative stability may be measured by relative settling times of each root or pair of roots. The settling time is inversely proportional to the location of roots of characteristic equation. If the root is located far away from the imaginary axis, then the transients dies out faster and so the relative stability of system will improve. The transient response and so the relative stability for various location of roots in s-plane are shown in fig 4.6.

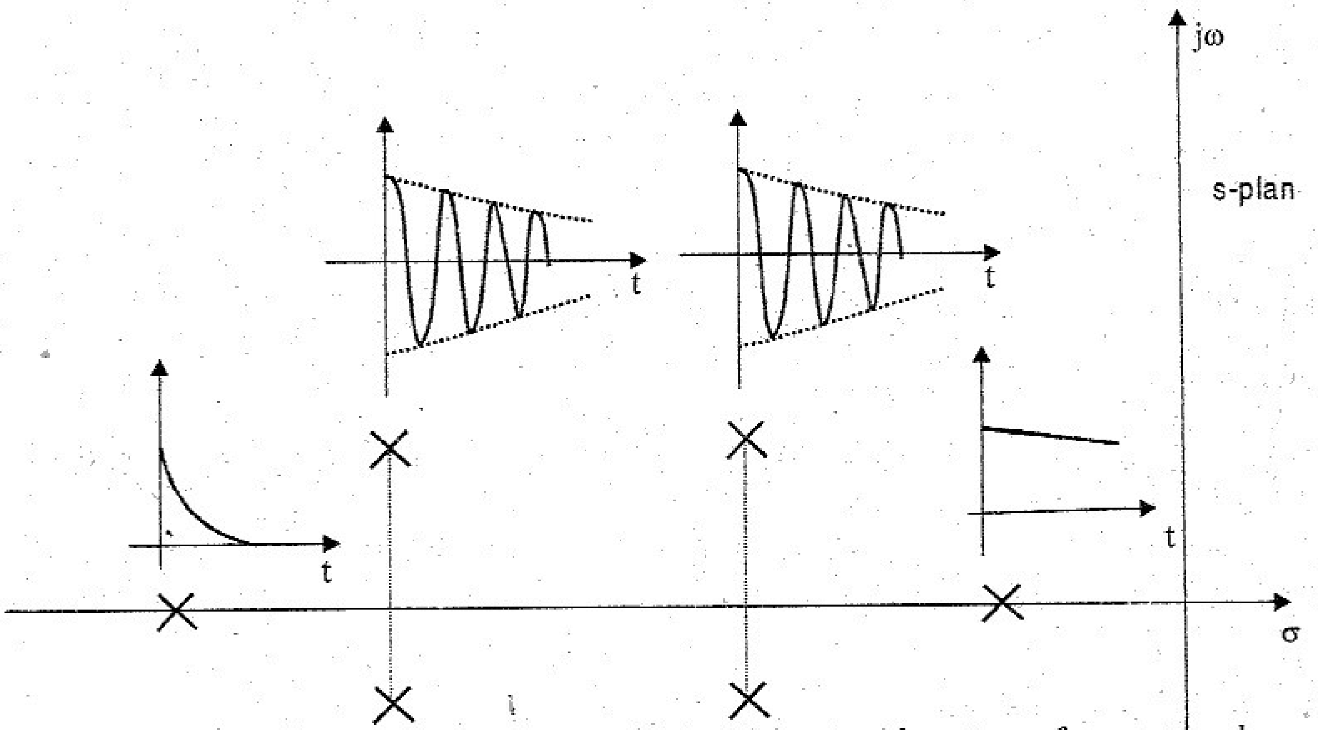


Fig 4.6 : Transient response and relative stability for various locations of roots on s-plane

In frequency domain the relative stability of a system can be studied from Nyquist plot. The relative stability of the system is given by closeness of polar plot to -1+j0 point. As the polar plot gets closer to -1+j0 point the system moves towards instability.

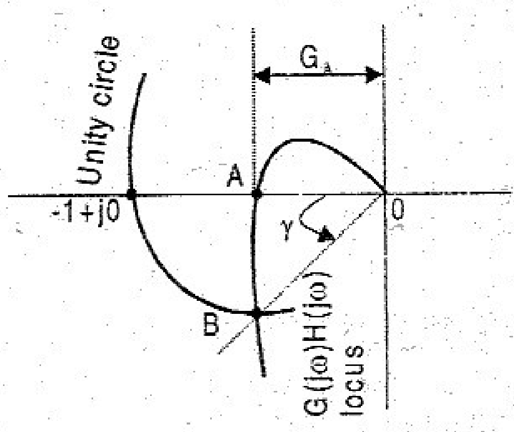


Fig : 4.7

The relative stability in frequency domain are quantitatively measured in terms of phase margin and gain margin. Consider a G(jω)H(jω) locus as shown in fig 4.7. Let this locus cross the real axis at point-**A** and a unit circle drawn with origin as centre cuts this locus at point-**B**. Let G_A be the magnitude of G(jω)H(jω) at point-A , and γ be the angle between negative real axis and line OB.

If the gain of the system is increased, then the locus will shift upwards and it may cross real axis at $-1+j0$ point. When the locus passes through $-1+j0$ point, $G_A \to 1$ and $\gamma \to 0$. Hence the closeness of G(jω) H(jω) locus to the critical point $-1+j0$ can be measured in terms of intercept G_A and angle γ. The value of G_A and γ are quantitative indications of relative stability. These values are used to define gain margin and phase margin as practical measures of relative stability.

The concepts of gain margin and phase margin are defined for open loop systems but from the values of gain margin and phase margin the stability of closed loop system can be judged.

4.7 GAIN MARGIN AND PHASE MARGIN

Gain margin is a factor by which the system gain can be increased to drive the system to the verge of instability. With reference to fig 4.7 the magnitude of G(jω)H(jω) is G_A when it crosses real axis and the phase corresponding to that point is $-180°$. The frequency corresponding to that point be ω_{pc}. If the gain of the system is increased by a factor $\frac{1}{G_A}$ then the magnitude at the frequency ω_{pc} will be, $G_A \times \frac{1}{G_A} = 1$. Now the G(jω)H(jω) locus will pass through $-1+j0$ point driving the system to the verge of instability. Hence the gain margin, K_g of the system may be defined as the reciprocal of the gain at which the phase angle is 180°. The frequency at which the phase angle is 180° is called phase crossover frequency.

Gain margin, $K_g = \left.\frac{1}{|G(j\omega)H(j\omega)|}\right|_{\omega=\omega_{pc}} = \frac{1}{G_A}$

Gain margin in db $= 20\log\frac{1}{|G(j\omega)H(j\omega)|} = 20\log\frac{1}{G_A} = -20\log G_A$

> ***Note :*** *Gain Margin in decibels is given by negative of db magnitude of $G(j\omega)$ at phase crossover frequency. Hence at ω_{pc}, if db magnitude is negative, then gain margin is positive and vice versa.*

The ***phase margin*** is defined as the amount of additional phase lag at gain crossover frequency required to bring the system to verge of instability. The frequency at which the magnitudes of $G(j\omega)H(j\omega)$ equals unity is called the gain crossover frequency, ω_{gc}. With reference to fig 4.7, the phase angle corresponding to the meeting point of unity circle and $G(j\omega)H(j\omega)$ locus is $-180° + \gamma$. Now with magnitude remaining unity, if an additional phase lag equal to γ is introduced then the net phase angle becomes $-180°$ and $G(j\omega)H(j\omega)$ locus will pass through $-1+j0$ point driving the system to the verge of instability. This additional phase lag γ is known as phase margin.

Let, $\phi_{gc} = \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$; Now $-180°+\gamma = \phi_{gc}$

$\therefore$ Phase margin, $\gamma = 180° + \phi_{gc}$

For stability of closed loop system the gain margin of open loop system should be greater than 1 or if it is expressed in db it should be positive and phase margin of open loop system should be positive.

EXAMPLE 4.19

The open loop transfer function of a unity feedback system is given by, $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$ Derive an expression for gain K in terms of T_1, T_2 and specified gain margin, K_g.

SOLUTION

Given that, $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$

Let $s = j\omega$.

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} = \frac{K}{j\omega(1+j\omega T_2 + j\omega T_1 - \omega^2 T_1 T_2)}$$

$$= \frac{K}{j\omega\left[1+j\omega(T_1+T_2)-\omega^2 T_1T_2\right]} = \frac{K}{j\omega - \omega^2(T_1+T_2) - j\omega^3 T_1 T_2}$$

$$= \frac{K}{-\omega^2(T_1+T_2) + j\omega(1-\omega^2 T_1 T_2)}$$

The gain margin, K_g is defined as the reciprocal of the magnitude of G(jω) at phase crossover frequency. At phase crossover frequency the magnitude is purely real. Hence at phase crossover frequency, ω_{pc}, the imaginary part of G(jω) is zero.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}\left(1-\omega_{pc}^2 T_1T_2\right) = 0 \quad \Rightarrow \quad 1-\omega_{pc}^2 T_1T_2 = 0 \quad \Rightarrow \quad -\omega_{pc}^2 T_1T_2 = -1$$

$$\therefore \omega_{pc} = \frac{1}{\sqrt{T_1T_2}}$$

At $\omega = \omega_{pc}$, the imaginary part is zero,

$$\therefore |G(j\omega)| = \left|\frac{K}{-\omega^2(T_1+T_2)}\right| = \frac{K}{\omega^2(T_1+T_2)}$$

$$\therefore \text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{K/\omega_{pc}^2(T_1+T_2)} = \frac{\omega_{pc}^2(T_1+T_2)}{K}$$

Put, $\omega_{pc}^2 = \frac{1}{T_1T_2}$ in the above equation.

$$\therefore K_g = \frac{\left(\frac{1}{T_1T_2}\right)(T_1+T_2)}{K} \quad \Rightarrow \quad K = \frac{1}{K_g}\,\frac{T_1+T_2}{T_1T_2} = \frac{1}{K_g}\left(\frac{T_1}{T_1T_2}+\frac{T_2}{T_1T_2}\right) \quad \Rightarrow \quad K = \frac{1}{K_g}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)$$

RESULT

The expression for gain K in terms of K_g, T_1 and T_2 is, $K = \frac{1}{K_g}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)$

EXAMPLE 4.20

Determine the Gain crossover frequency, Phase crossover frequency, Gain margin and Phase margin of a system with open loop transfer function, $G(s) = \frac{1}{s(1+2s)(1+s)}$.

SOLUTION

(i) To find phase crossover frequency and gain margin

Given that, $G(s) = \frac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j2\omega)\,(1+j\omega)} = \frac{1}{j\omega\left(1+j\omega+j2\omega-2\omega^2\right)} = \frac{1}{j\omega\,(j3\omega+1-2\omega^2)} = \frac{1}{-3\omega^2+j\omega\,(1-2\omega^2)}$$

At phase crossover frequency the imaginary part of G(jω) is zero. Hence put $\omega = \omega_{pc}$ in imaginary part and equate to zero to solve for ω_{pc}.

$$\therefore \omega_{pc}\left(1-2\omega_{pc}^2\right) = 0$$

$$\text{since } \omega_{pc} \neq 0, \; 1-2\omega_{pc}^2 = 0 \;\Rightarrow\; -2\omega_{pc}^2 = -1 \;\Rightarrow\; \omega_{pc}^2 = \frac{1}{2} \;\Rightarrow\; \omega_{pc} = \frac{1}{\sqrt{2}} = 0.707 \text{ rad/sec}$$

The gain margin, K_g is defined as reciprocal of magnitude of G(jω) at phase cross over frequency.

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{|1/-3\omega^2|_{\omega=\omega_{pc}}} = 3\omega_{pc}^2 = 3\times 0.707^2 = 1.5$$

Gain margin in db = 20 log K_g = 20 log 1.5 = 3.5 db

(ii) To find gain crossover frequency and phase margin

Given that, $G(s) = \frac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$.

$$\therefore\ G(j\omega) = \frac{1}{j\omega(1+j2\omega)(1+j\omega)} = \frac{1}{\omega\angle 90^\circ \sqrt{1+4\omega^2}\angle \tan^{-1}2\omega \sqrt{1+\omega^2}\angle \tan^{-1}\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}2\omega - \tan^{-1}\omega$$

At gain crossover frequency, ω_{gc} the magnitude of $G(j\omega)$ is unity.

$$\therefore \text{At } \omega = \omega_{gc}, \qquad |G(j\omega)| = \frac{1}{\omega_{gc}\sqrt{1+4\omega_{gc}^2}\sqrt{1+\omega_{gc}^2}} = 1$$

Solving the above equation for ω_{gc} will be tedious. Hence by trial and error find the root of the above equation.

$$\text{When } \omega = 1, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = \frac{1}{1\sqrt{1+4}\sqrt{1+1}} = 0.3$$

$$\text{When } \omega = 0.5, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = \frac{1}{0.5\sqrt{1+4\times0.5^2}\sqrt{1+0.5^2}} = 1.26$$

From the two calculations shown above, we can conclude that the unity magnitude will occur for a frequency between 0.5 and 1.0.

$$\text{When } \omega = 0.6, \quad |G(j\omega)| = \frac{1}{0.6\sqrt{1+4\times0.6^2}\sqrt{1+0.6^2}} = 0.915$$

$$\text{When } \omega = 0.57, \quad |G(j\omega)| = \frac{1}{0.57\sqrt{1+4\times0.57^2}\sqrt{1+0.57^2}} = 1.005$$

Let $\omega = 0.57$ be the gain crossover frequency, since for this value of ω the magnitude of $G(j\omega)$ is approximately equal to one.

$\therefore$ Gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec.

Let the phase of $G(j\omega)$ at ω_{gc} be ϕ_{gc}.

$$\text{At } \omega = \omega_{gc} = 0.57, \qquad \phi_{gc} = -90^\circ - \tan^{-1}2\omega - \tan^{-1}\omega$$

$$= -90^\circ - \tan^{-1}(2\times0.57) - \tan^{-1}0.57 = -168^\circ$$

$$\therefore \text{Phase margin, } \gamma = 180^\circ + \phi_{gc} = 180^\circ - 168^\circ = 12^\circ$$

RESULT

(a) The phase crossover frequency, $\omega_{pc} = 0.707$ rad/sec

(b) The gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec

(c) The gain margin, $K_g = 1.5$

The gain margin in db = 3.5 db

(d) The phase margin, $\gamma = 12^\circ$

EXAMPLE 4.21

The open loop transfer function of a system is $G(s) = \dfrac{K}{s\,(1+0.1s)\,(1+s)}$.

(i) Determine the value of K so that gain margin is 6 db.

(ii) Determine the value of K so that phase margin is 40°.

SOLUTION

(i) To find K for specified gain margin

Given that, $G(s) = \dfrac{K}{s\,(1+0.1s)\,(1+s)}$

Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{K}{j\omega\,(1+j0.1\omega)\,(1+j\omega)} = \frac{K}{j\omega\,(1+j1.1\omega - 0.1\omega^2)} = \frac{K}{-1.1\omega^2 + j\omega\,(1-0.1\omega^2)}$$

At phase crossover frequency ω_{pc}, the $G(j\omega)$ is real and so equate the imaginary part to zero to solve for ω_{pc}.

$$\text{At } \omega = \omega_{pc}, \qquad \omega_{pc}(1-0.1\omega_{pc}^2) = 0 \quad \Rightarrow \quad 1-0.1\omega_{pc}^2 = 0 \quad \Rightarrow \quad -0.1\omega_{pc}^2 = -1$$

$$\therefore \omega_{pc} = \frac{1}{\sqrt{0.1}} = 3.162 \text{ rad / sec}$$

$$\therefore |G(j\omega)|_{\omega=\omega_{pc}} = \left|\frac{K}{-1.1\omega^2}\right|_{\omega=\omega_{pc}} = \frac{K}{1.1\times 3.162^2} = 0.0909K$$

Given that gain margin = 6db, $\quad \therefore 20 \log K_g = 6 \quad \Rightarrow \quad \log K_g = \dfrac{6}{20}$

$$\therefore \text{ Gain margin, } K_g = 10^{\frac{6}{20}} = 1.9953$$

By definition of gain margin,

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$$

$$\therefore 1.9953 = \frac{1}{0.0909K}$$

$$\therefore K = \frac{1}{0.0909 \times 1.9953} = 5.5135$$

(ii) To find K for specified phase margin

Given that, $G(s) = \dfrac{K}{s\,(1+0.1s)\,(1+s)}$. Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{K}{j\omega\,(1+j0.1\omega)\,(1+j\omega)} = \frac{K}{\omega\angle 90^\circ \;\sqrt{1+(0.1\omega)^2}\;\angle\tan^{-1}0.1\omega \;\sqrt{1+\omega^2}\;\angle\tan^{-1}\omega}$$

$$|G(j\omega)| = \frac{K}{\omega\sqrt{1+0.01\omega^2}\;\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.1\omega - \tan^{-1}\omega$$

Let, ω_{gc} = Gain crossover frequency

$\phi_{gc} = \angle G(j\omega)$ at $\omega = \omega_{gc}$.

At $\omega = \omega_{gc}$, $\quad \phi_{gc} = \angle G(j\omega)|_{\omega=\omega_{gc}} = -90 - \tan^{-1}0.1\,\omega_{gc} - \tan^{-1}\omega_{gc}$

By definition of phase margin,

Phase margin, $\gamma = 180° + \phi_{gc}$

The required phase margin is 40°, $\therefore \gamma = 40°$

$$\therefore\ 40° = 180° - 90° - \tan^{-1}0.1\omega_{gc} - \tan^{-1}\omega_{gc} \quad \Rightarrow \quad \tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 180° - 90° - 40°$$

$$\therefore\ \tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 50°$$

On taking tan on either side we get,

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\tan\left[\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc}\right] = \tan 50°$$

$$\frac{\tan\tan^{-1}0.1\omega_{gc} + \tan\tan^{-1}\omega_{gc}}{1 - \tan\tan^{-1}0.1\omega_{gc} \times \tan\tan^{-1}\omega_{gc}} = \tan 50° \Rightarrow \frac{0.1\omega_{gc} + \omega_{gc}}{1 - 0.1\omega_{gc} \times \omega_{gc}} = 1.192 \Rightarrow \frac{1.1\omega_{gc}}{1 - 0.1\omega_{gc}^2} = 1.192$$

On cross multipling the above equation we get,

$$1.1\,\omega_{gc} = 1.192\,(1 - 0.1\omega_{gc}^2) \quad \Rightarrow \quad 0.1192\omega_{gc}^2 + 1.1\omega_{gc} - 1.192 = 0$$

$$\therefore\ \omega_{gc}^2 + \frac{1.1}{0.1192}\,\omega_{gc} - \frac{1.192}{0.1192} = 0 \quad \Rightarrow \quad \omega_{gc}^2 + 9.228\,\omega_{gc} - 10 = 0$$

$$\therefore \omega_{gc} = \frac{-9.228 \pm \sqrt{9.228^2 + 4 \times 10}}{2} = \frac{-9.228 \pm 11.1873}{2}$$

On taking positive value we get,

$$\omega_{gc} = \frac{-9.228 + 11.1873}{2} = 0.98 \text{ rad / sec}.$$

$$\text{At } \omega = \omega_{gc}, \quad |G(j\omega)| = 1 \quad ; \quad |G(j\omega)|_{\omega=\omega_{gc}} = \frac{K}{\omega_{gc}\sqrt{1 + 0.01\omega_{gc}^2}\sqrt{1 + \omega_{gc}^2}} = 1$$

$$\therefore\ K = \omega_{gc}\sqrt{1 + 0.01\omega_{gc}^2}\sqrt{1 + \omega_{gc}^2} = 0.98\sqrt{1 + 0.01 \times 0.98^2}\sqrt{1 + 0.98^2} = 1.3787$$

RESULT

For a gain margin of 6 db, $\quad K = 5.5135$

For a phase margin of 40°, $\quad K = 1.3787$

4.8 ROOT LOCUS

The root locus technique was introduced by **W.R.Evans** in 1948 for the analysis of control systems. The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

Consider the open loop transfer function of system $G(s) = \dfrac{K}{s\,(s + p_1)\,(s + p_2)}$

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}{1 + \dfrac{K}{s\,(s+p_1)\,(s+p_2)}} = \frac{K}{s\,(s+p_1)\,(s+p_2) + K}$$

The denominator polynomial of C(s)/R(s) is the characteristic equation of the system. The characteristic equation is given by,

$$s\,(s + p_1)\,(s + p_2) + K = 0\,.$$

The roots of characteristic equation is a function of open loop gain K. [In other words the roots of characteristic equation depend on open loop gain K]. When the gain K is varied from 0 to ∞, the roots of characteristic equation will take different values. When K = 0, the roots are given by open loop poles. When $K \rightarrow \infty$, the roots will take the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called ***root loci*** (or the path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus).

> ***Note :*** *In general the roots of characteristic equation can be varied by varying any other system parameter other than gain.*

In general the closed loop transfer function of system with multiple loops is obtained from the signal flow graph of the system using Mason's gain formula.

$$\frac{C(s)}{R(s)} = T(s) = \frac{1}{\Delta}\sum_k P_k \Delta_k \qquad \textit{(Refer chapter 1 section 1.12)}$$

The determinant, Δ is the denominator polynominal of C(s)/R(s). The characteristic equation of the system is given by , $\Delta = 0$

For the single loop system shown in fig 4.8

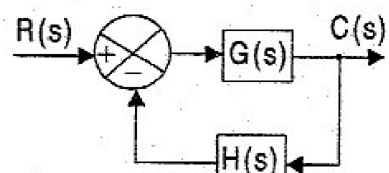


Fig 4.8

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The Characteristic equation is,

$$1 + G(s)\,H(s) = 0$$

$$\therefore\ G(s)\,H(s) = -1 \qquad .....(4.22)$$

From equation (4.22) it can be concluded that the roots of the characteristic equation occur only for those values of s for which, $G(s)H(s) = -1$.

The equation (4.22) can be converted to two Evans conditions given below,

$$|G(s)H(s)| = 1 \qquad .....(4.23)$$

$$\angle G(s)H(s) = \pm 180^\circ (2q + 1), \qquad \text{where } q = 0,1,2,3, ... \qquad .....(4.24)$$

The equation (4.23) is called magnitude criterion and equation (4.24) is called angle criterion.

The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s,

$|G(s)\ H(s)| = 1$.

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s, $\angle G(s)\,H(s)$ is equal to an odd multiple of 180°.

The function G(s)H(s) can be expressed as a ratio of two polynomials in s as shown below.

$$G(s)H(s) = K\,\frac{(s+z_1)\,(s+z_2)\,(s+z_3).....}{(s+p_1)\,(s+p_2)\,(s+p_3).....} \qquad .....(4.25)$$

$$\therefore |G(s)H(s)| = K\,\frac{|s+z_1|\times|s+z_2|\times|s+z_3|.....}{|s+p_1|\times|s+p_2|\times|s+p_3|....} = K\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|}$$

where, m = Number of zeros of loop transfer function.

n = Number of poles of loop transfer function.

The magnitude criterion states that $|G(s)H(s)| = 1$.

$$\therefore K\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = 1 \quad \text{or} \quad K = \frac{\prod_{i=1}^{n}|s+p_i|}{\prod_{i=1}^{m}|s+z_i|} \qquad .....(4.26)$$

The open-loop gain K corresponding to a point $s = s_a$ on root locus can be calculated using equation (4.26). It can be shown that $|s + p_i|$ is equal to the length of vector drawn from $s = p_i$ to $s = s_a$ and $|s + z_i|$ is equal to the length of vector drawn from $s = z_i$ to $s = s_a$. Hence the equation K can be written as,

$$\boxed{K = \frac{\text{Product of length of vector from open loop poles to the point } s = s_a}{\text{Product of length of vectors from open loop zeros to the point } s = s_a}}$$

From equation (4.25),

$$\angle G(s)H(s) = \angle(s+z_1) + \angle(s+z_2) + \angle(s+z_3) + ...- \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)....$$

$$= \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$$

where, m = Number of zeros of loop transfer function.

n = Number of poles of loop transfer function.

The angle criterion states that $\angle G(s)H(s) = \pm 180°\ (2q + 1)$

$$\therefore \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = \pm 180°(2q+1) \qquad .....(4.27)$$

The equations (4.27) can be used to check whether a point $s = s_a$ is a point on root locus or not. It can be shown that $\angle(s+p_i)$ is equal to the angle of vector drawn from $s = p_i$ to $s = s_a$ and $\angle(s+z_i)$ is equal to the angle of vector drawn from $s = z_i$ to $s = s_a$. Hence equation (5.27) can be written as

$$\begin{pmatrix}\text{Sum of angles of vector}\\ \text{from open loop zeros}\\ \text{to the point } s = s_a\end{pmatrix} - \begin{pmatrix}\text{Sum of angles of vector}\\ \text{from open loop poles}\\ \text{to the point } s = s_a\end{pmatrix} = \pm 180°\ (2q+1)$$

CONSTRUCTION OF ROOT LOCUS

The exact root locus is sketched by trial and error procedure. In this method, the poles and zeros of G(s)H(s) are located on the s-plane on a graph sheet and a trial point $s = s_a$ is selected. Determine the angles of vectors drawn from poles and zeros to the trial point. From the angle criterion, determine the angle to be contributed by these vectors to make the trial point as a point on root locus. Shift the trial point suitably so that the angle criterion is satisfied.

A number of points are determined using the above procedure. Join the points by a smooth curve which is the root locus. The value of K for a particular root can be obtained from the magnitude criterion.

The trial and error procedure for sketching root locus is tedious. A set of rules have been developed to reduce the task involved in sketching root locus and to develop a quick approximate sketch. From the approximate sketch, a more accurate root locus can be obtained by a few trials.

RULES FOR CONSTRUCTION OF ROOT LOCUS

Rule 1 : The root locus is symmetrical about the real axis.

Rule 2 : Each branch of the root locus originates from an open-loop pole corresponding to K = 0 and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K = \infty$. The number of branches of the root locus terminating on infinity is equal to $n-m$, (i.e., the number of open loop poles minus the number of finite zeros)

Rule 3 : Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.

Rule 4 : The $n-m$ root locus branches that tend to infinity, do so along straight line asymptotes making angles with the real axis given by,

$$\phi_A = \frac{180°(2q+1)}{n-m} \;;\quad q = 0,1,2,\ldots, n-m.$$

Rule 5 : The point of intersection of the asymptotes with the real axis is at $s = \sigma_A$ where,

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Rule 6 : The breakaway and breakin points of the root locus are determined from the roots of the equation dK/ds = 0. If r numbers of branches of root locus meet at a point, then they break away at an angle of $\pm 180°/r$.

Rule 7 : The angle of departure from a complex open-loop pole is given by,

$$\phi_p = \pm 180°\,(2q+1) + \phi \;;\qquad q = 0, 1, 2, \ldots$$

where ϕ is the net angle contribution at the pole by all other open loop poles and zeros. Similarly the angle of arrival at a complex open loop zero is given by,

$$\phi_z = \pm 180°\,(2q+1) + \phi \;;\qquad q = 0, 1, 2, \ldots$$

where ϕ is the net angle contribution at the zero by all other open-loop poles and zeros.

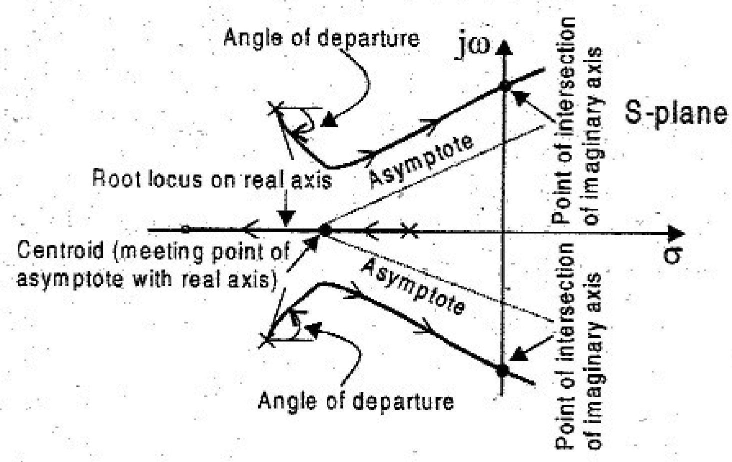
Rule 8 : The points of intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion. Alternatively they can be evaluated by letting $s = j\omega$ in the characteristic equation and equating the real part and imaginary part to zero, to solve for ω and K. The values of ω are the intersection points on imaginary axis and K is the value of gain at the intersection points.

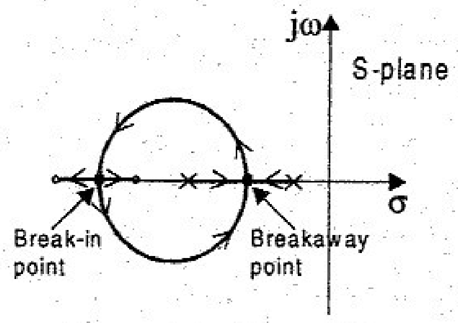
Rule 9 : The open-loop gain K at any point $s = s_a$ on the root locus is given by,

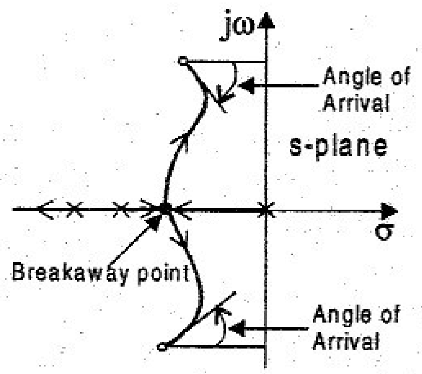
$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|} = \frac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a}$$

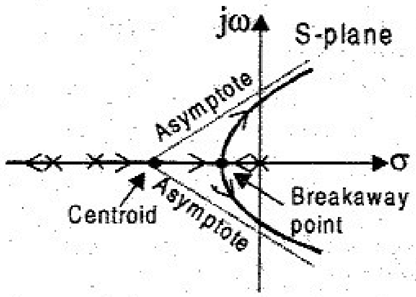
> ***Note :*** *The length of vector should be measured to scale. If there is no finite zero then the product of vector lengths from zeros is equal to 1.*

TYPICAL SKETCHES OF ROOT LOCUS PLOTS









PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Locate the poles and zeros of G(s)H(s) on the s-plane. The root locus branch starts from open loop poles and terminates at zeros.

Step 2 : Determine the root locus on real axis.

Step 3 : Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step 4 : Find the breakaway and breakin points.

Step 5 : If there is a complex pole then determine the angle of departure from the complex pole. If there is a complex zero then determine the angle of arrival at the complex zero.

Step 6 : Find the points where the root loci may cross the imaginary axis.

Step 7 : Take a series of test points in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test points by smooth curve.

Step 8 : The value of gain K at any point on the locus can be determined from magnitude condition.

The value of K at a point $s = s_a$, is given by,

$$K = \frac{\text{product of length of vectors from poles to the point, } s = s_a}{\text{product of length of vectors from finite zeros to the point, } s = s_a}$$

> *Note : When there is no finite zero, the denominator is taken as unity. The length of vectors should be measured to scale.*

EXPLANATION FOR THE VARIOUS STEPS IN THE PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Location of poles and zeros

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real and imaginary axis.

The poles are marked by cross **"X"** and zeros are marked by small circle **"o"**. The number of root locus branches is equal to number of poles of open loop transfer function. The origin of a root locus is at a pole and the end is at a zero.

Let, n = number of poles
m = number of finite zeros

Now, **m** root locus branches ends at finite zeros. The remaining **n–m** root locus branches will end at zeros at infinity.

Step 2 : Root locus on real axis

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3 : Angles of asymptotes and centroid

If n is number of poles and m is number of finite zeros, then $n-m$ root locus branches will terminate at zeros at infinity.

These n–m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. Hence number of asymptotes is equal to number of root locus branches going to infinity. The angles of asymptotes and the centroid are given by the following formulae.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

where, $q = 0, 1, 2, 3, \ldots\ldots(n-m)$

$$\text{Centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Step 4 : Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis between 2 poles then there exist a breakaway point. If there is a root locus on real axis between 2 zeros then there exist a breakin point. If there is a root locus on real axis between pole and zero then there may be or may not be breakaway or breakin point.

Let the characteristic equation be in the form,

$$B(s) + K\,A(s) = 0$$

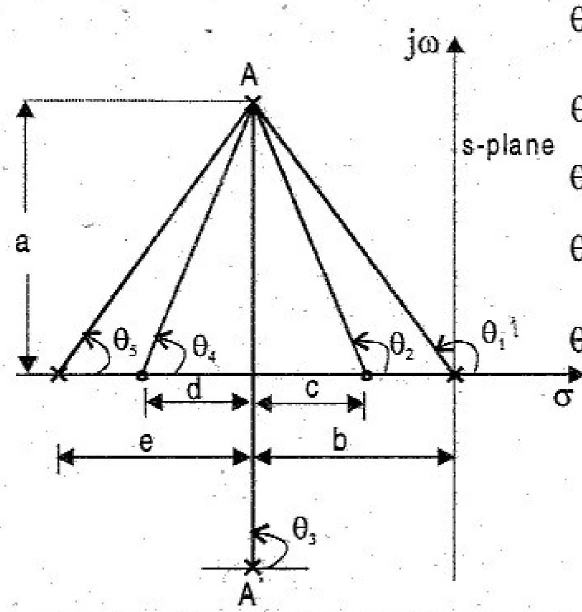
$$\therefore K = \frac{-B(s)}{A(s)}$$

The breakaway and breakin point is given by roots of the equation dK/ds = 0. The roots of dK/ds = 0 are actual breakaway or breakin point provided for this value of root, the gain K should be positive and real.

Step 5 : Angle of Departure and angle of arrival

$$\left.\begin{matrix}\text{Angle of Departure}\\ \text{(from a complex pole A)}\end{matrix}\right\} = 180° - \begin{pmatrix}\text{Sum of angles of vector to the}\\ \text{complex pole A from other poles}\end{pmatrix} + \begin{pmatrix}\text{Sum of angles of vectors to the}\\ \text{complex pole A from zeros}\end{pmatrix}$$

Note : *The angles can be calculated as shown in fig 4.9 or they can be measured using protractor.*



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 4.9 : Calculation of angle of departure

Example:

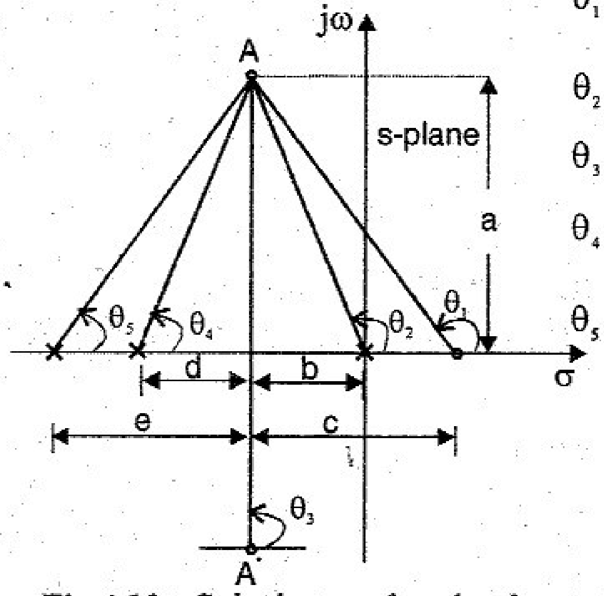
Consider the two complex conjugate poles A and A* shown in fig 4.9.(If poles are complex then they exist only as conjugate pairs)

$$\left.\begin{matrix}\text{Angle of departure}\\ \text{at pole A}\end{matrix}\right\} = 180° - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

$$\left.\begin{matrix}\text{Angle of departure}\\ \text{at pole A *}\end{matrix}\right\} = -[\text{Angle of departure at pole A}]$$

$$\left.\begin{matrix}\text{Angle of arrival at a}\\ \text{complex zero A}\end{matrix}\right\} = 180° - \left(\begin{matrix}\text{Sum of angles of vectors to the}\\ \text{complex zero A from all other zeros}\end{matrix}\right) + \left(\begin{matrix}\text{Sum of angles of vectors to the}\\ \text{complex zero A from poles}\end{matrix}\right)$$

***Note :** The angles can be calculated as shown in fig 4.10 or they can be measured using protractor.*



$$\theta_1 = 180° - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180° - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90°$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 4.10 : *Calculation of angle of arrival*

Example:

Consider the two complex conjugate zeros B and B* as shown in fig 4.10.(If zeros are complex then they exist only as conjugate pairs)

$$\left.\begin{matrix}\text{Angle of arrival}\\ \text{at zero B}\end{matrix}\right\} = 180° - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

$$\left.\begin{matrix}\text{Angle of arrival}\\ \text{at zero B*}\end{matrix}\right\} = -[\text{Angle of arrival at zero B}]$$

Step 6 : Point of intersection of root locus with imaginary axis

The point where the root loci intersects the imaginary axis can be found by following three methods.

1. By Routh Hurwitz array.
2. By trial and error approach.
3. Letting $s = j\omega$ in the characteristic equation and separate the real part and imaginary part. Two equations are obtained : *one by equating real part to zero and the other by equating imaginary part to zero.* Solve the two equations for ω and **K**. The values of ω gives the points where the root locus crosses imaginary axis. The value of K gives the value of gain K at there crossing points. Also this value of K is the limiting value of K for stability of the system.

Step 7 : Test points and root locus

Choose a test point. Using a protractor roughly estimate the angles of vectors drawn to this point and adjust the point to satisfy angle criterion. Repeat the procedure for few more test points. Sketch the root locus from the knowledge of typical sketches and the informations obtained in steps 1 through 6.

> *Note : In practice the approximate root locus can be sketched from the informations obtained in steps 1 through 6 and from the knowledge of typical sketches of root locus.*

DETERMINATION OF OPEN LOOP GAIN FOR A SPECIFIED DAMPING OF THE DOMINANT ROOTS

The dominant pole is a pair of complex conjugate pole which decides the transient response of the system. In higher order systems the dominant poles are given by the poles which are very close to origin, provided all other poles are lying far away from the dominant poles. The poles which are far away from the origin will have less effect on the transient response of the system.

The transfer function of higher order systems can be approximated to a second order transfer function. The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The dominant poles, s_d and $s_d{}^*$, are given by the roots of quadratic factor, $s^2+2\zeta\omega_n s+\omega_n{}^2 = 0$.

$$\therefore s_d = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The dominant pole can be plotted on the s-plane as shown in fig 4.11.

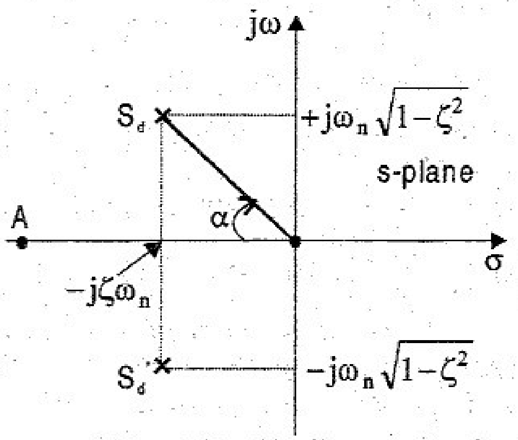


Fig 4.11 : Dominant pole, s_d

In fig 4.11, the right angle triangle OAP,

$$\cos\alpha = \frac{\zeta\omega_n}{\omega_n} = \zeta, \qquad \therefore \alpha = \cos^{-1}\zeta$$

To fix a dominant pole on root locus, draw a line at an angle of $\cos^{-1}\zeta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

Let, K_{sd} be the value of gain at dominant pole s_d,

$$\text{Now,}\quad K_{sd} = \frac{\text{Product of length of vectors from open loop poles to dominant pole}}{\text{Product of length of vectors from open loop zeros to dominant pole}}$$

Importance of root locus

The root locus technique is an important tool in designing control systems with desired performance characteristics. The desired performance of the system can be achieved by adjusting the location of its closed loop poles in the s-plane by varying one or more system parameters.

The root locus can be plotted in the s-plane by varying a system parameter (usually gain, K) over the complete range of values. The roots corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus.

The root locus technique is also used for stability analysis. Using root locus the range of values of K, for a stable system can be determined. It is also easier to study the relative stability of the system from the knowledge of location of closed loop poles. The dominant roots are used to estimate the damping ratio and natural frequency of oscillation of the system. From ζ and ω_n the time domain specifications can be calculated.

EXAMPLE 4.22

A unity feedback control system has an open loop transfer function, $G(s) = \dfrac{K}{s\,(s^2+4s+13)}$. Sketch the root locus.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are lying at $s = 0, -2+j3$ and $-2-j3$.

Let us denote the poles as P_1, P_2, and P_3.

Here, $P_1 = 0$, $P_2 = -2+j3$ and $P_3 = -2-j3$.

The poles are marked by X (cross) as shown in fig 4.22.1.

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig 4.22.1.

> ***Note :*** *For the given transfer function one root locus branch will start at the pole at the origin and meet the zero at infinity through the negative real axis.*

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ\,(2q+1)}{n-m} \quad ; \quad q = 0, 1, \ldots\ldots\ldots n-m$$

Here $n = 3$, and $m = 0$. $\quad \therefore q = 0, 1, 2, 3$.

When $q = 0$, $\quad \text{Angles} = \pm\dfrac{180^\circ}{3} = \pm 60^\circ$

When $q = 1$, $\quad \text{Angles} = \pm\dfrac{180^\circ \times 3}{3} = \pm 180^\circ$

When $q = 2$, $\quad \text{Angles} = \pm\dfrac{180^\circ \times 5}{3} = \pm 300^\circ = \mp 60^\circ$

When $q = 3$, $\quad \text{Angles} = \pm\dfrac{180^\circ \times 7}{3} = \pm 420^\circ = \pm 60^\circ$

> ***Note :*** *It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0 - 2 + j3 - 2 - j3 - 0}{3} = \frac{-4}{3} = -1.33$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 4.22.1.

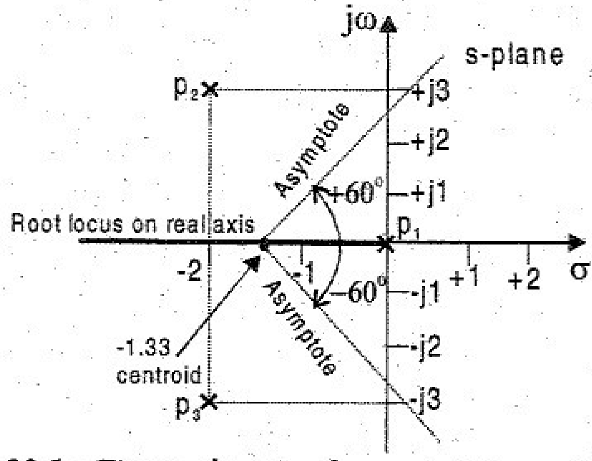


Fig 4.22.1 : Figure showing the asymptote, root locus on real axis and location of poles and centroid

Step 4 : To find the breakaway and breakin points

The closed loop transfer function $\left.\right\}$ $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K}{s\,(s^2+4s+13)}}{1+\dfrac{K}{s\,(s^2+4s+13)}} = \dfrac{K}{s\,(s^2+4s+13)+K}$

The characteristic equation is, $s(s^2+4s+13)+K=0$

$$\therefore \quad s^3+4s^2+13s+K=0 \quad \Rightarrow \quad K=-s^3-4s^2-13s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\dfrac{dK}{ds}=0$

$$\therefore \; -(3s^2+8s+13)=0 \quad \Rightarrow \quad (3s^2+8s+13)=0$$

$$\therefore s = \frac{-8 \pm \sqrt{8^2-4\times 13\times 3}}{2\times 3} = -1.33 \pm j1.6$$

Check for K : When, $s=-1.33+j1.6$, the value of K is given by,

$$K=-(s^3+4s^2+13s)=-[(-1.33+j1.6)^3+4\,(-1.33+j1.6)^2+13\,(-1.33+j1.6)]$$

$$\neq \text{positive and real.}$$

Also it can be shown that when $s=-1.33-j1.6$ the value of K is not equal to real and positive.

Since the values of K for, $s=-1.33\pm j1.6$, are not real and positive, these points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step 5 : To find the angle of departure

Let us consider the complex pole p_2 shown in fig 4.22.2. Draw vectors from all other poles to the pole p_2 as shown in fig 4.22.2. Let the angles of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(3/2) = 123.7°$; $\theta_2 = 90°$

Angle of departure from the complex pole $p_2 = 180° - (\theta_1+\theta_2)$

$$= 180° - (123.7° + 90°)$$

$$= -33.7°$$

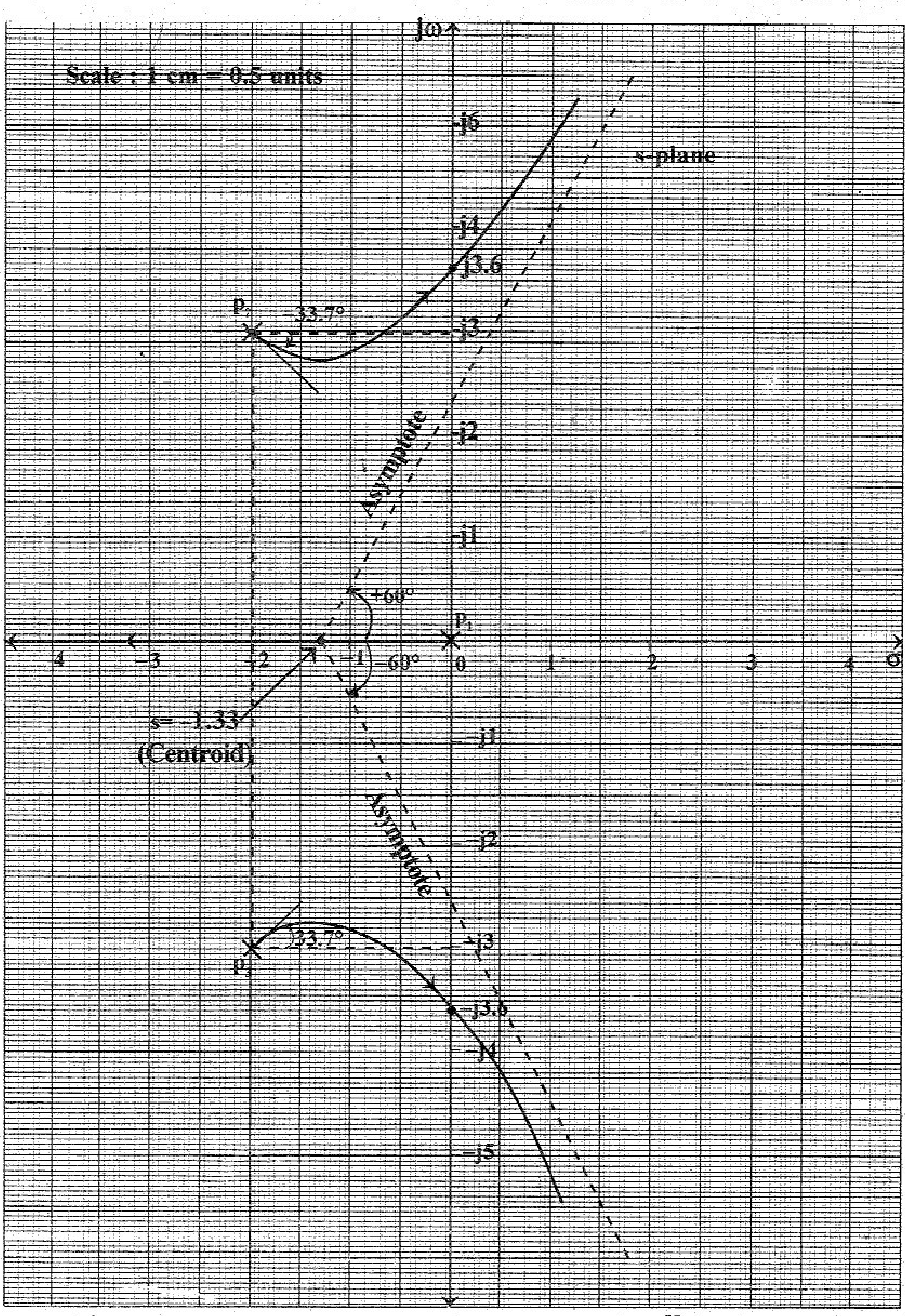


Fig 4.22.3. : Root locus sketch of, $1+G(s)=1+\dfrac{K}{s\,(s^2+4s+13)}$.

The angle of departure at complex pole p_3 is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole $p_3 = +33.7°$

Mark the angles of departure at complex poles using protractor.

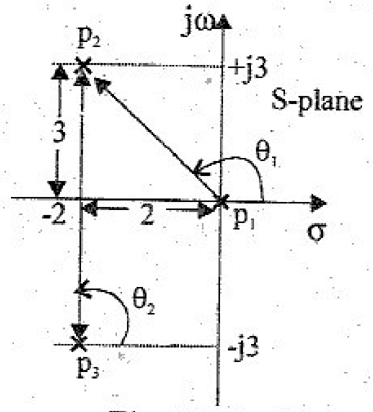


Fig 4.22.2

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \quad \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

On equating imaginary part to zero, we get,

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

$$\omega^2 = 13 \quad \Rightarrow \quad \omega = \pm\sqrt{13} = \pm 3.6$$

On equating real part to zero, we get,

$$-4\omega^2 + K = 0$$

$$K = 4\omega^2$$

$$= 4 \times 13 = 52$$

The crossing point of root locus is $\pm$ j3.6. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 4.22.3. The root locus has three branches one branch starts at the pole at origin and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at complex poles (along the angle of departure), crosses the imaginary axis at $\pm$j3.6 and travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 4.23

Sketch the root locus of the system whose open loop transfer function is, $G(s) = \dfrac{K}{s\,(s+2)\,(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+2)(s+4) = 0$.

$\therefore$ The poles are lying at, $s = 0, -2, -4$.

Let us denote the poles as p_1, p_2, and p_3.

Here, $p_1 = 0$, $p_2 = -2$, $p_3 = -4$.

The poles are marked by X(cross) as shown in fig 4.23.1.

Step 2 : To find the root locus on real axis

There are three poles on the real axis.

Choose a test point on real axis between s = 0 and s = –2. To the right of this point the total number of real poles and zeros is one, which is an odd number. Hence the real axis between s = 0 and s = –2 will be a part of root locus.

Choose a test point on real axis between s = –2 and s = –4. To the right of this point, the total number of real poles and zeros is two which is an even number. Hence the real axis between s = –2 and s = –4 will not be a part of root locus.

Choose a test point on real axis to the left of s = –4. To the right of this point, the total number of real poles and zeros is three, which is an odd number. Hence the entire negative real axis from s = –4 to –∞ will be a part of root locus.

The root locus on real axis are shown as bold lines in fig 4.23.1.

Step 3 : To find asymptotes and centroid

Since there are three poles the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ\ (2q+1)}{n-m} \quad ; \qquad q = 0, 1, 2, \ldots\ldots n-m.$$

Here, n = 3 and m = 0. $\quad \therefore q = 0, 1, 2, 3$.

When $q = 0$, $\quad \text{Angles} = \pm\dfrac{180^\circ}{3} = \pm 60^\circ$

When $q = 1$, $\quad \text{Angles} = \pm\dfrac{180^\circ \times 3}{3} = \pm 180^\circ$

> ***Note :** It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2-4-0}{3} = -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 4.23.1.

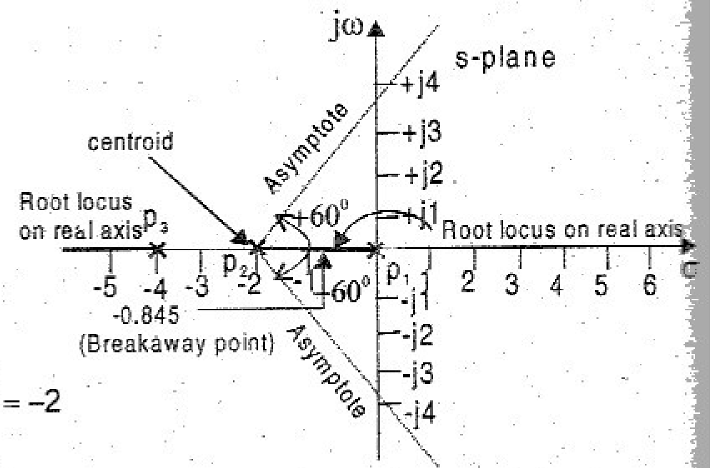


Fig 4.23.1 : Figure showing the asymptotes, root locus on real axis and location of poles, centroid ,and breakaway points.

Step 4 : To find the breakaway and breakin points

$$\left.\begin{matrix}\text{The closed loop}\\ \text{transfer function}\end{matrix}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+2)\,(s+4)}}{1+\dfrac{K}{s\,(s+2)\,(s+4)}} = \frac{K}{s\,(s+2)\,(s+4)\,+\,K}$$

The characteristic equation is given by,

$$s\,(s+2)\,(s+4)+K=0 \quad\Rightarrow\quad s\,(s^2+6s+8)+K=0 \quad\Rightarrow\quad s^3+6s^2+8s+K=0$$

$$\therefore K = -s^3-6s^2-8s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+12s+8)$$

$$\text{Put} \quad \frac{dK}{ds} = 0$$

$$\therefore -(3s^2+12s+8) = 0 \quad\Rightarrow\quad (3s^2+12s+8) = 0$$

$$s = \frac{-12 \pm \sqrt{12^2-4\times3\times8}}{2\times3} = -0.845 \quad \text{or} \quad -3.154$$

Check for K : When s = −0.845, the value of K is given by,

$$K = -[(-0.845)^3+6(-0.845)^2+8(-0.845)] = 3.08$$

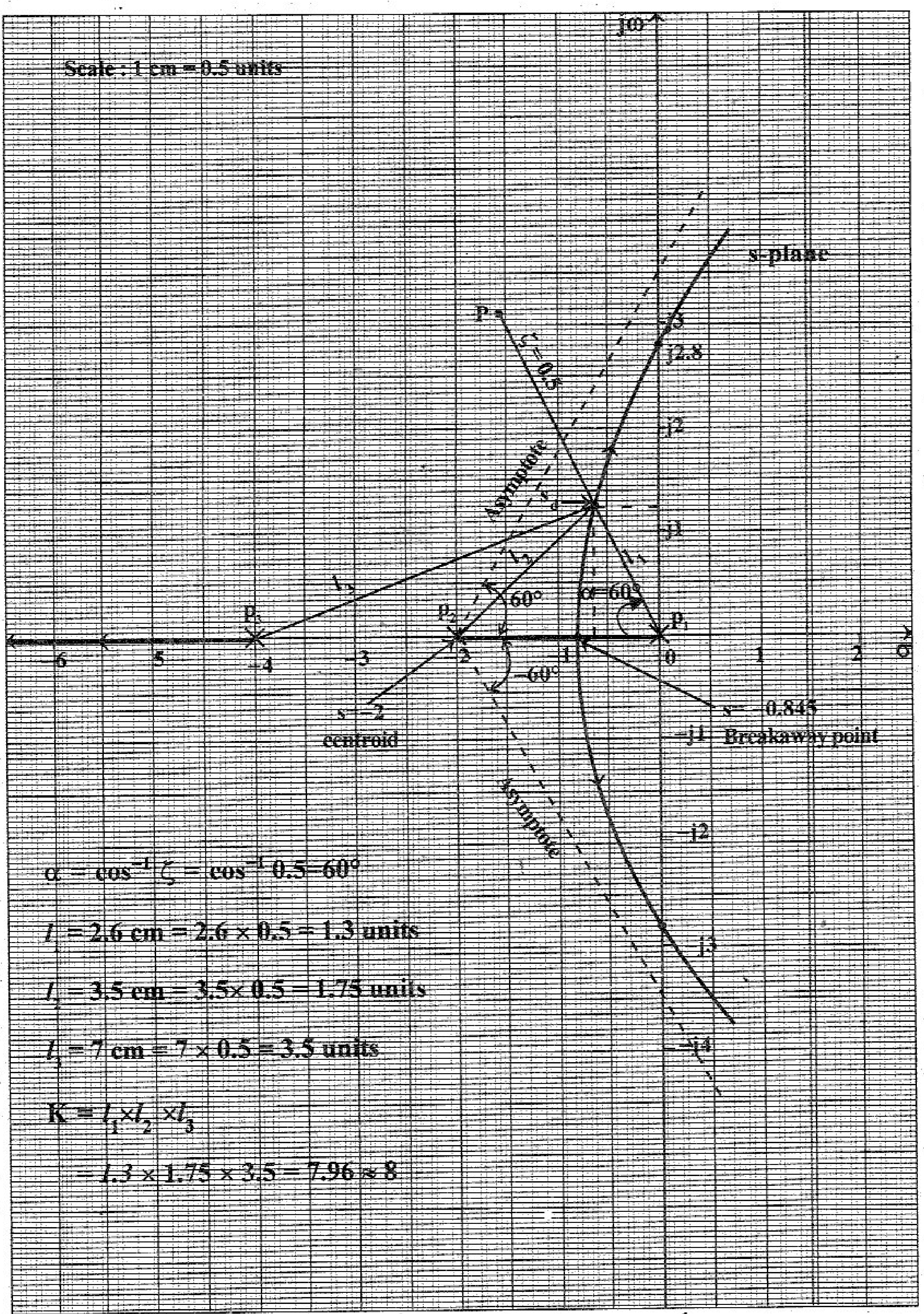
Since K, is positive and real for, s = −0.845, this point is actual breakaway point.

When s = −3.154, the value of K is given by,

$$K = -\,[(-3.154)^3+6(-3.154)^2+8(-3.154)] = -3.08$$

Since K, is negative for, s = −3.154, this is not a actual breakaway point.

The breakaway point is marked on the negative real axis as shown in fig 4.23.1.



$\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60°$

$l_1 = 2.6$ cm $= 2.6 \times 0.5 = 1.3$ units

$l_2 = 3.5$ cm $= 3.5 \times 0.5 = 1.75$ units

$l_3 = 7$ cm $= 7 \times 0.5 = 3.5$ units

$K = l_1 \times l_2 \times l_3$

$= 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$

Fig 4.23.2. : Root locus sketch of, $1 + G(s) = 1 + \dfrac{1}{s(s+2)(s+4)}$.

Step 5 : To find angle of departure

Since there are no complex pole or zero, we need not find angle of departure or arrival.

Step 6 : To find the crossing point of imaginary axis

The characteristic equation is given by,

$$s^3 + 6s^2 + 8s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 - 6\omega^2 + j8\omega + K = 0$$

Equating imaginary part to zero

$$-j\omega^3 + j8\omega = 0$$

$$-j\omega^3 = -j8\omega$$

$$\omega^2 = 8 \quad \Rightarrow \quad \omega = \pm\sqrt{8} = \pm 2.8$$

Equating real part to zero

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2 = 6 \times 8 = 48$$

The crossing point of root locus is $\pm j2.8$. The value of K corresponding to this point is K = 48. (This is the limiting value of K for the stability of the system).

The complete root locus sketch is shown in fig 4.23.2. The root locus has three branches. One branch starts at the pole at $s = -4$ and travel through negative real axis to meet the zero at infinity. The other two root locus branches starts at $s = 0$ and $s = -2$ and travel through negative real axis, breakaway from real axis at $s = -0.845$, then crosses imaginary axis at $s = \pm j2.8$ and travel parallel to asymptotes to meet the zeros at infinity

To find the value of K corresponding to $\zeta = 0.5$

Given that $\zeta = 0.5$

Let $\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60°$

Draw a line OP, such that the angle between line OP and negative real axis is 60° ($\alpha = 60°$) as shown in fig 4.23.2. The meeting point of the line OP and root locus gives the dominant pole, s_d.

Let K_{sd} be value of K corresponding to the point $s = s_d$

$$K_{sd} = \frac{\text{Product of length of vector from all poles to the point, } s = s_d}{\text{Product of length of vector from all zeros to the point, } s = s_d}$$

$$= \frac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$$

Note : The length of vectors are measured to scale.

EXAMPLE 4.24

The open loop transfer function of a unity feedback system is given by, $G(s) = \dfrac{K(s+9)}{s(s^2+4s+11)}$. Sketch the root locus of the system.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equations, $(s^2 + 4s + 11) = 0$.

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 11}}{2} = -2 \pm j2.64$

$\therefore$ The poles are lying at, s = 0, –2 + j2.64, –2 –j2.64

The zeros are lying at, s = –9 and infinity.

Let us denote the poles as p_1, p_2, p_3 finite zero by z_1.

Here, $p_1 = 0$, $p_2 = -2+j2.64$, $p_3 = -2-j2.64$ and $z_1 = -9$.

The poles are marked by X(cross) and zeros by "o" (circle) as shown in fig 4.24.1.

Step 2 : To find the root locus on real axis.

One pole and one zero lie on real axis.

Choose a test point to the left of s = 0, then to the right of this point, the total number of poles and zeros is one which is an odd number. Hence the portion of real axis from s = 0 to s = –9 will be a part of root locus.

If we choose a test point to the left of s = –9 then to the right of this point, the total number of poles and zeros is two, which is an even number. Hence the real axis from s = –9 to –∞ will not be a part of root locus.

The root locus on real axis is shown as a bold line in fig 4.24.1.

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. One root locus branch starts at the pole at origin and travel along negative real axis to meet the zero at s = –9. The other two root locus branches meet the zeros at infinity. The number of asymptotes required are two.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}; \qquad q = 0, 1, 2, ...n-m.$$

Here, n = 3 and m = 0. $\therefore$ q = 0, 1, 2, 3 .

When q = 0, $\text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$

When q = 1, $\text{Angles} = \pm\frac{180^\circ \times 3}{2} = \pm 270^\circ = \mp 90^\circ$

When q = 2, $\text{Angles} = \pm\frac{180^\circ \times 5}{2} = \pm 450^\circ = \pm 90^\circ$

> ***Note :*** *It is enough if you calculate the required number of angles. Here it is given by first two values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j2.64-2-j2.64-(-9)}{2} = 2.5$$

The centroid is marked and from the centroid, the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown 4.24.1.

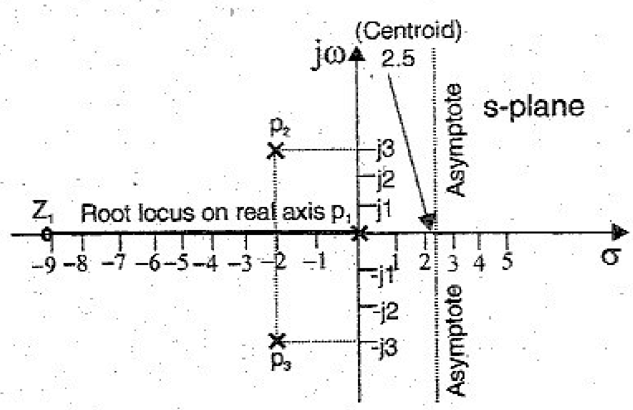


Fig 4.24.1 : Figure showing the asymptotes, root locus on real axis and location of poles, zero and centroid

Step 4 : To find the breakaway and breakin points

From the location of poles and zero and from the knowledge of typical sketches of root locus, it can be concluded that there is no possibility of breakaway or breakin points.

Step 5 : To find the angle of departure

Let us consider the complex pole p_2 as shown in fig 4.24.2. Draw vectors from all other poles and zero to the pole p_2 as shown in fig 4.24.2. Let the angles of these vectors be θ_1, θ_2 and θ_3.

$$\text{Here, } \theta_1 = 180° - \tan^{-1}\frac{2.64}{2} = 127.1°$$

$$\theta_2 = 90°$$

$$\theta_3 = \tan^{-1}\frac{2.64}{7} = 20.7°$$

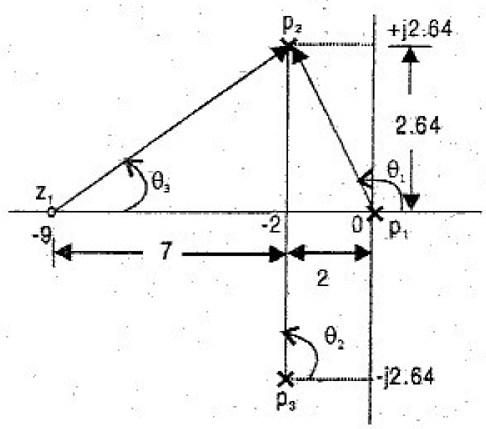


Fig 4.24.2

$$\left.\begin{array}{l}\text{Angle of departure from}\\ \text{the complex pole } p_2\end{array}\right\} = 180° - (\theta_1 + \theta_2) + \theta_3$$

$$= 180° - (127.1° + 90°) + 20.7° = -16.4°$$

The angle of departure at the complex pole p_3 is negative of the angle of departure at complex pole p_3.

$\therefore$ Angle of departure at pole $p_3 = -(-16.4) = +16.4°$

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point of imaginary axis

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K(s+9)}{s\,(s^2+4s+11)}}{1+\dfrac{K\,(s+9)}{s\,(s^2+4s+11)}} = \frac{K(s+9)}{s\,(s^2+4s+11)+K(s+9)}$$

The characteristic equation is the denominator polynomial of C(s)/R(s).

$\therefore$ The characteristic equation is,

$$s\,(s^2+4s+11)+K\,(s+9)=0 \;\Rightarrow\; (s^3+4s^2+11s)+Ks+9K=0$$

put $s = j\omega$

$$(j\omega)^3+4(j\omega)^2+11(j\omega)+K(j\omega)+9K=0 \quad\Rightarrow\quad -j\omega^3-4\omega^2+j11\omega+jK\omega+9K=0$$

On equating imaginary part to zero,

$$-j\omega^3+j11\omega+jK\omega=0 \Rightarrow -j\omega^3=-j11\omega-jK\omega$$

$$\therefore\; \omega^2=11+K$$

Put $K=8.8$, $\therefore \omega^2 = 11+8.8 = 19.8$

$$\omega = \pm\sqrt{19.8} = \pm 4.4$$

On equating real part to zero,

$$-4\omega^2+9K=0 \quad\Rightarrow\quad 9K=4\omega^2$$

Put, $\omega^2=11+K$ $\;\therefore 9K = 4(11+K) = 44+4K$

$$\therefore 9K-4K=44$$

$$\therefore 5K=44 \quad\Rightarrow\quad K=\frac{44}{5}=8.8$$

The crossing point of root locus is $\pm j4.4$. The value of K at this crossing point is K = 8.8 (This is the limiting value of K for the stability of the system).

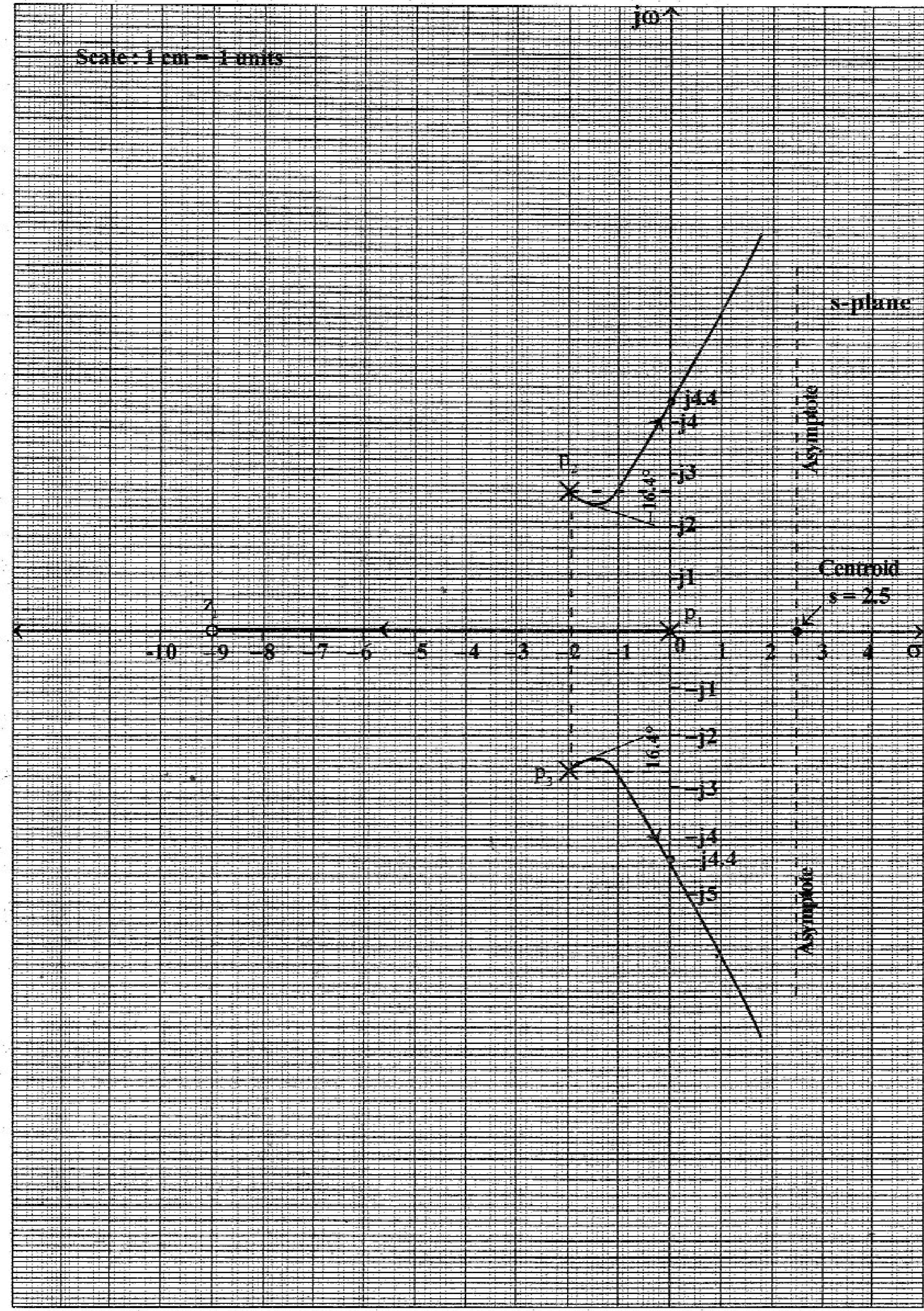


Fig 4.24.3. : Root locus sketch for, $1+G(s)=1+\dfrac{K(s+9)}{s(s^2+4s+11)}$.

The complete root locus sketch is shown in fig 4.24.3. The root locus has three branches. One branch starts at pole at origin and travel through negative real axis to meet the zero at s = –9.

The other two root locus branches starts at complex poles (along the angle of departure) crosses the imaginary axis at ±j4.4 and travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 4.25

Sketch the root locus for the unity feedback system whose open loop transfer function is,

$$G(s)\,H(s) = \frac{K}{s\,(s+4)\,(s^2+4s+20)}.$$

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s+4)\,(s^2+4s+20) = 0$.

$$\text{The roots of the quadratic are, } s = \frac{-4 \pm \sqrt{4^2 - 4\times1\times20}}{2} = -2 \pm j4$$

$\therefore$ The poles are lying at, $s = 0, -4, -2+j4$ and $-2-j4$.

The zeros are lying at infinity.

Let us denote the poles as p_1, p_2, p_3 and p_4.

Here, $p_1 = 0$, $p_2 = -4$, $p_3 = -2+j4$, and $p_4 = -2-j4$.

The poles are marked by X (cross) as shown in fig 4.25.1.

Step 2 : To find root locus on real axis

There are two poles on the real axis. Choose a test point on real axis between s = 0 and s = –4. To the right of this point, the total number of real poles is one which is an odd number. Hence the real axis between s = 0 and s = –4 will be a part of root locus. Choose a test point to the left of s = –4, now to the right of this test point the total number of poles and zeros is two which is even number. Hence the real axis from s = –4 to s = –∞ will not be a part of root locus. The root locus on real axis is shown as a bold line in fig 4.25.1.

Step 3 : To find angles of asymptotes and centroid

Since there are four poles, the number of root locus branches are four. There is no finite zero. Hence all the four root locus branches ends at zeros at infinity. Hence the number of asymptotes required is four.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ\,(2q+1)}{n-m}; \qquad q = 0, 1, 2, \ldots\ldots n-m.$$

Here, n = 3 and m = 0. $\quad \therefore q = 0, 1, 2, 3$.

Note : It is enough if you calculate the required number of angles. Here it is given by first four values of angles. The remaining will be repetiticns of the previous values.

$$\text{When } q = 0, \quad \text{Angles} = \pm\frac{180^\circ}{4} = \pm45^\circ$$

$$\text{When } q = 1, \quad \text{Angles} = \pm\frac{180^\circ \times 3}{4} = \pm135^\circ$$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-4-2+j4-2-j4-0}{4-0} = \frac{-8}{4} = -2.$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 4.25.1.

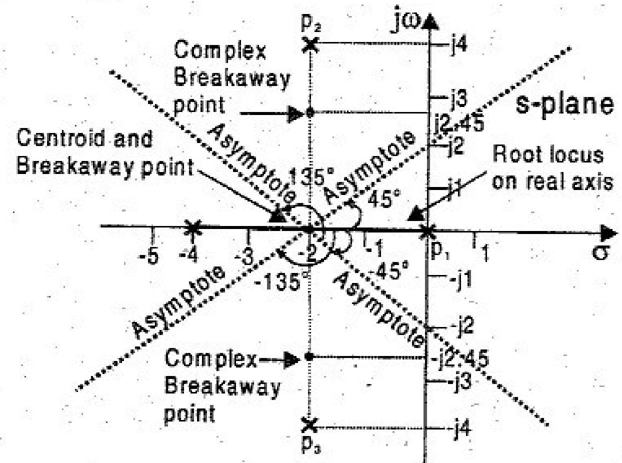


Fig 4.25.1 : Figure showing the asymptotes, root locus on real axis and location of poles, centroid and breakaway points.

Step 4 : To find the breakaway and breakin point

The closed loop transfer function $\left.\right\} \dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K}{s\,(s+4)\,(s^2+4s+20)}}{1+\dfrac{K}{s\,(s+4)\,(s^2+4s+20)}} = \dfrac{K}{s\,(s+4)\,(s^2+4s+20)+K}$

The characteristic equation is, $s\,(s+4)\,(s^2+4s+20)+K=0.$

$$\therefore K = -s\,(s+4)\,(s^2+4s+20) = -(s^2+4s)\,(s^2+4s+20)$$
$$\therefore K = -(s^4+8s^3+36s^2+80s)$$

On differentiating the equation of K with respect to s we get, $\dfrac{dK}{ds} = -(4s^3+24s^2+72s+80)$.

To find the real root of $\dfrac{dK}{ds}$ = 0 by Lin's method.

The first trial divisor is chosen as the last two terms of the polynomial

Ist trial

Trial divisor $= 18s+20 = s+\dfrac{20}{18} = s+1.11$

$$\begin{array}{r|l} & s^2+4.89s+12.57 \\ \hline s+1.11 & s^3+6s^2+18s+20 \\ & \underline{s^3+1.11s^2} \\ & \quad 4.89s^2+18s \\ & \quad \underline{4.89s^2+5.43s} \end{array}$$

Next trial divisor $\rightarrow 12.57s+20$

$$\begin{array}{r} 12.57s+13.95 \\ \hline 6.05 \\ \hline \end{array}$$

IInd trial

Trial divisor $= 12.57s+20 = s+\dfrac{20}{12.57} = s+1.59$

$$\begin{array}{r|l} & s^2+4.41s+11 \\ \hline s+1.59 & s^3+6s^2+18s+20 \\ & \underline{s^3+1.59s^2} \\ & \quad 4.41s^2+18s \\ & \quad \underline{4.41s^2+7s} \end{array}$$

Next trial divisor $\rightarrow 11s+20$

$$\begin{array}{r} 11s+17.49 \\ \hline 2.51 \\ \hline \end{array}$$

IIIrd trial

Trial divisor = 11s + 20

$$= s + \frac{20}{11} = s + 1.82$$

$$\begin{array}{r|l}
 & s^2 + 4.18s + 10.4 \\
\hline
s + 1.82 & s^3 + 6s^2 + 18s + 20 \\
 & s^3 + 1.82s^2 \\
\hline
 & \quad 4.18s^2 + 18s + 20 \\
 & \quad 4.18s^2 + 7.6s \\
\hline
 & \qquad\quad 10.4s + 20 \\
 & \qquad\quad 10.4s + 18.9 \\
\hline
 & \qquad\qquad\quad 1.1
\end{array}$$

Since the remainder converge for every trial, let us approximate the root to s = –2. On dividing the polynomial by s + 2, we found that (s+2) is a divisor of the polynomial.

$$\begin{array}{r|l}
 & s^2 + 4s + 10 \\
\hline
s + 2 & s^3 + 6s^2 + 18s + 20 \\
 & s^3 + 2s^2 \\
\hline
 & \quad 4s^2 + 18s \\
 & \quad 4s^2 + 8s \\
\hline
 & \qquad 10s + 20 \\
 & \qquad 10s + 20 \\
\hline
 & \qquad\quad 0
\end{array}$$

Put, $\frac{dK}{ds} = 0$. $\quad \therefore (-4s^3 + 24s^2 + 72s + 80) = 0 \quad \Rightarrow \quad 4s^3 + 24s^2 + 72s + 80 = 0.$

On dividing by 4 we get, $s^3 + 6s^2 + 18s + 20 = 0.$

The equation $s^3 + 6s^2 + 18s + 20 = 0$ will have atleast one real root. By trial and error, the real root is found to be $s = -2$. *(Refer Appendix II for Lin's method.)*

The polynomial, $(s^3 + 6s^2 + 18s + 20) = 0$, can be expressed as,

$$s^3 + 6s^2 + 18s + 20 = (s + 2)(s^2 + 4s + 10) = 0$$

The root of the quadratic, $s^2 + 4s + 10 = 0$, are given by,

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 10}}{2} = -2 \pm j2.45$$

Check for K : When, $s = -2$, $K = -(s^4 + 8s^3 + 36s^2 + 80s) = -\left[(-2)^4 + 8 \times (-2)^3 + 36 \times (-2)^2 + 80 \times (-2)\right]$

$$= -[-64] = 64$$

When, $s = -2 \pm j2.45 = 3.16\angle \pm 129°$

$$K = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$= -(3.16\angle \pm 129°)^4 + 8 \times (3.16\angle \pm 129°)^3 + 36 \times (3.16\angle \pm 129°)^2 + 80 \times 3.16\angle \pm 129°$$

$$= -[99.7\angle \pm 156° + 252.4\angle \pm 27° + 359.5\angle \pm 258° + 252.8\angle \pm 129°]$$

For positive values of angles,

$$K = -[-91 + j40 + 225 + j115 - 75 - j351 - 159 + j196] = -[-100] = 100$$

For negative values of angles,

$$K = -[-91 - j40 + 225 + j115 - 75 + j351 - 159 - j196] = -[-100] = 100$$

For all the roots of the equation dK/ds = 0, the value of K is positive and real. Hence all the three roots are actual breakaway points. The breakaway points are shown in fig 4.25.1.

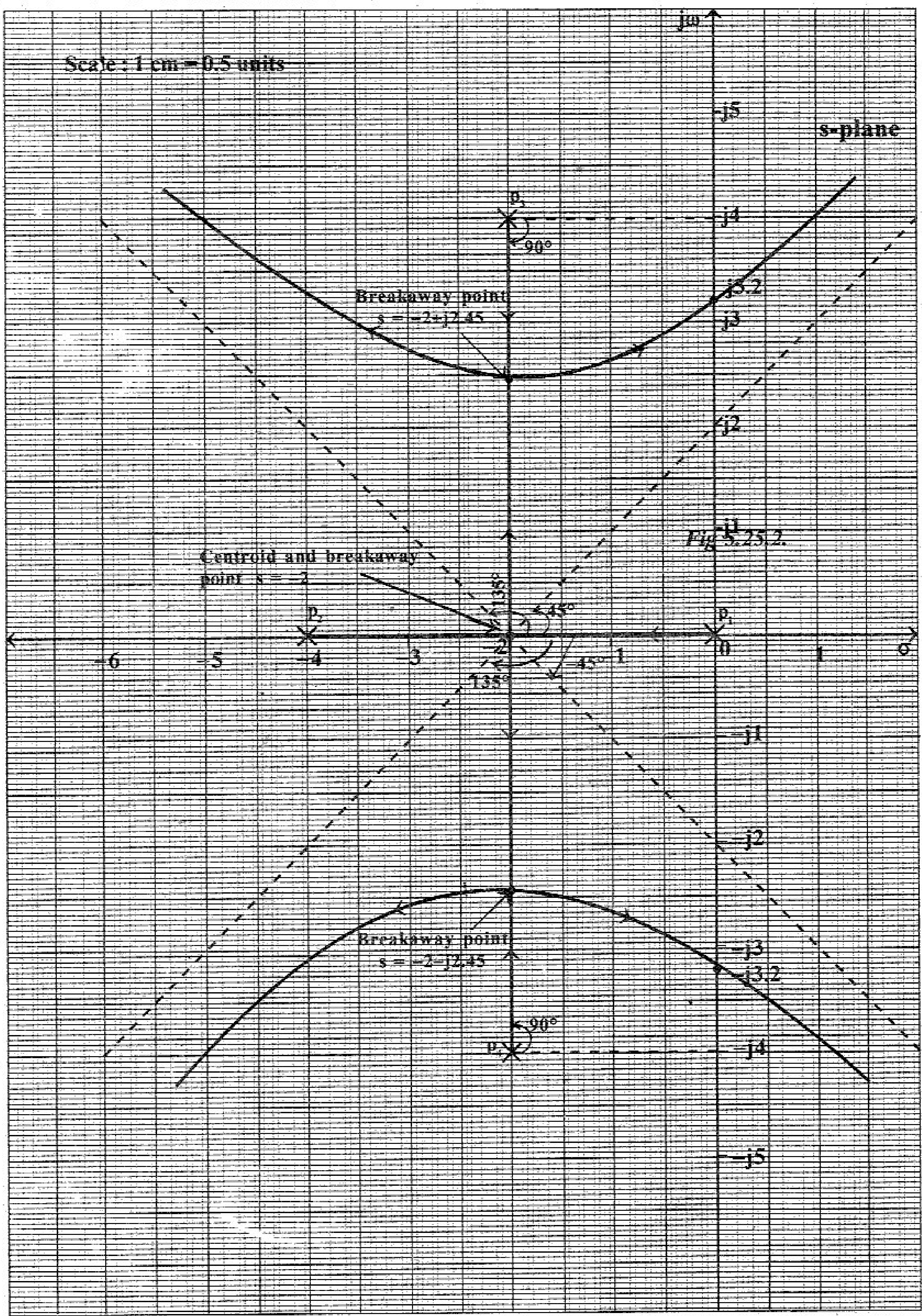


Fig 4.25.3. : Root locus sketch of, $1+G(s) = 1+\dfrac{K}{s\,(s+4)\,(s^2+4s+20)}$.

Step 5 : To find angle of departure

Let us consider the complex pole p_3 shown in fig 4.25.2. Draw vectors from all other poles to the pole p_3 as shown in fig 4.25.2. Let angles of these vectors be θ_1, θ_2 and θ_3.

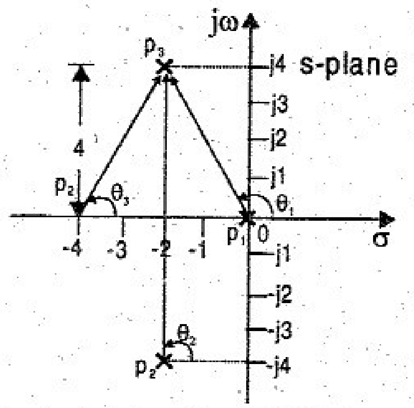


Fig 4.25.2

Here,

$$\theta_1 = 180° - \tan^{-1}\frac{4}{2} = 117°$$

$$\theta_2 = 90°$$

$$\theta_3 = \tan^{-1}\frac{4}{2} = 63°$$

$$\left.\begin{array}{l}\text{Angle of departure}\\ \text{from complex pole } p_3\end{array}\right\} = 180° - (\theta_1 + \theta_2 + \theta_3)$$

$$= 180° - (117° + 90° + 63°) = -90°$$

The angle of departure at complex pole p_4 is negative of the angle of departure at complex pole p_3.

$\therefore$ Angle of departure from complex pole p_4 = +90°

Mark the angles of departure at complex poles using protractor.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by, $s^4 + 8s^3 + 36s^2 + 80s + K = 0$.

put $s = j\omega$,

$$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + K = 0.$$

$$\omega^4 - j8\omega^3 - 36\omega^2 + j80\omega + K = 0.$$

On equating imaginary part to zero,

$$-j8\omega^3 + j80\omega = 0$$

$$-j8\omega^3 = -j80\omega$$

$$\omega^2 = 10$$

$$\omega = \pm\sqrt{10} = \pm 3.2$$

On equating real part to zero,

$$\omega^4 - 36\omega^2 + K = 0$$

$$K = -\omega^4 + 36\omega^2$$

Put $\omega^2 = 10$

$$\therefore K = -(10)^2 + (36 \times 10) = 260.$$

The crossing point of root locus is ±j3.2. The value of K at this crossing point is K = 260. (This is the limiting value of K for stability).

The complete root locus is sketched as shown in fig 4.25.3. The root locus has four branches. All the root locus branches goes to infinity along the asymptotic lines to meet the zeros at infinity.

EXAMPLE 4.26

Sketch root locus for the unity feedback system whose open loop transfer function is,

$$G(s)\,H(s) = \frac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}.$$

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+1)(s+5) = 0$ and the zeros are the roots of the equation, $(s+1.5) = 0$.

The poles are lying at, s = 0, –1, –5.

The zeros are lying at, s = –1.5 and infinity.

Let us denote poles by, p_1, p_2, p_3 and finite zero by z_1.

Here, $p_1 = 0$, $p_2 = -1$, $p_3 = -5$ and $z_1 = -1.5$.

The poles are marked by X(cross) and zeros by "o" (circle) as shown in fig 4.26.1.

Step 2 : To find root locus on real axis

The segment of real axis between s = 0 and s = −1 and the segment of real axis between s = −1.5 and s = −5 will be a part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros. The root locus on real axis are shown as bold lines in fig 4.26.1.

Step 3 : To find angles of asymptotes and centroid

Since there are three poles, the number of root locus branches are three. There is one finite zero, so one root locus branch will end at finite zero. The other two branches will meet the zeros at infinity. Hence the number of asymptotes required is two.

$$\text{Angles of asymptotes} = \frac{\pm 180^{\circ}\,(2q+1)}{n-m}\,; \qquad q = 0,\ 1,\ 2,\ ....n-m.$$

Here, n = 3 and m = 1. $\therefore$ q = 0, 1, 2 .

$$\text{When } q = 0, \quad \text{Angles} = \pm\frac{180^{\circ}}{2} = \pm 90^{\circ}$$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-1-5-(-1.5)}{2} = -2.25$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 4.26.1.

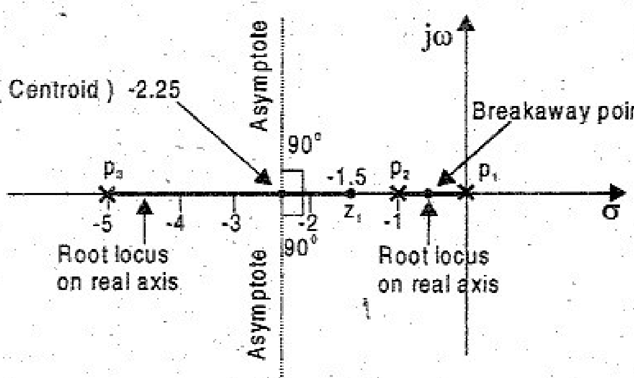


Fig 4.26.1 : Figure showing the asymptotes, root locus on real axis and location of poles, zeros, centroid and breakaway points.

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}}{1+\dfrac{K\,(s+1.5)}{s\,(s+1)\,(s+5)}} = \frac{K\,(s+1.5)}{s\,(s+1)\,(s+5)+K(s+1.5)}$$

The characteristic equation is, $s\,(s+1)\,(s+5) + K\,(s+1.5) = 0$

$$\therefore K = \frac{-s\,(s+1)\,(s+5)}{s+1.5} = \frac{-s\,(s^2+6s+5)}{s+1.5} = \frac{-(s^3+6s^2+5s)}{s+1.5}$$

On differentiating K with respect to s we get,

$$\frac{dK}{ds}=\frac{-(3s^2+12s+5)\,(s+1.5)-\left[-(s^3+6s^2+5s)\right](1)}{(s+1.5)^2}$$

$$=\frac{-3s^3-4.5s^2-12s^2-18s-5s-7.5+s^3+6s^2+5s}{(s+1.5)^2}$$

$$=\frac{-2s^3-10.5s^2-18s-7.5}{(s+1.5)^2}=\frac{-2(s^3+5.25s^2+9s+3.75)}{(s+1.5)^2}$$

For $\frac{dK}{ds}=0$, the numerator should be zero.

$\therefore\ s^3+5.25s^2+9s+3.75=0$

The third order polynomial will have one real root. The real root of the above polynomial can be determined by Lin's method. *(Refer Appendix II).*

To find the real root of, $s^3+5.25s^2+9s+3.75=0$, by Lin's method

The last two terms of the polynomial are chosen as Ist trial divisor.

Ist trial

Ist Trial divisor $=9s+3.75=s+\frac{3.75}{9}=s+0.42$

$$\begin{array}{r|l} & s^2+4.83s+6.97 \\ s+0.42 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.42s^2 \\ \hline & \quad 4.83s^2+9s \\ & \quad 4.83s^2+2.03s \\ \hline \text{IInd trial divisor} \rightarrow & \qquad 6.97s+3.75 \\ & \qquad 6.97s+2.93 \\ \hline & \qquad\qquad 0.82 \end{array}$$

IInd trial

IInd Trial divisor $=6.97s+3.75=s+\frac{3.75}{6.97}=s+0.54$

$$\begin{array}{r|l} & s^2+4.71s+6.46 \\ s+0.54 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.54s^2 \\ \hline & \quad 4.71s^2+9s \\ & \quad 4.71s^2+2.54s \\ \hline \text{IInd trial divisor} \rightarrow & \qquad 6.46s+3.75 \\ & \qquad 6.46s+3.49 \\ \hline & \qquad\qquad 0.26 \end{array}$$

IIIrd trial

IIIrd Trial divisor $=6.46s+3.75=s+\frac{3.75}{6.46}=s+0.58$

$$\begin{array}{r|l} & s^2+4.67s+6.3 \\ s+0.58 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.58s^2 \\ \hline & \quad 4.67s^2+9s \\ & \quad 4.67s^2+2.7s \\ \hline \text{IVth trial divisor} \rightarrow & \qquad 6.3s+3.75 \\ & \qquad 6.3s+3.65 \\ \hline & \qquad\qquad 0.1 \end{array}$$

IVth trial

IVth Trial divisor $=6.3s+3.75=s+\frac{3.75}{6.3}=s+0.6$

$$\begin{array}{r|l} & s^2+4.65s+6.2 \\ s+0.6 & s^3+5.25s^2+9s+3.75 \\ & s^3+0.6s^2 \\ \hline & \quad 4.65s^2+9s \\ & \quad 4.65s^2+2.8s \\ \hline & \qquad 6.2s+3.75 \\ & \qquad 6.2s+3.72 \\ \hline & \qquad\qquad 0.03 \end{array}$$

On neglecting the small value of 0.03, one of the root of the polynomial is, s = -0.6.

The polynomial, $s^3+5.25s^2+9s+3.75=0$, can be expressed,

$$s^3+5.25s^2+9s+3.75=(s+0.6)\,(s^2+4.65s+6.2)=0.$$

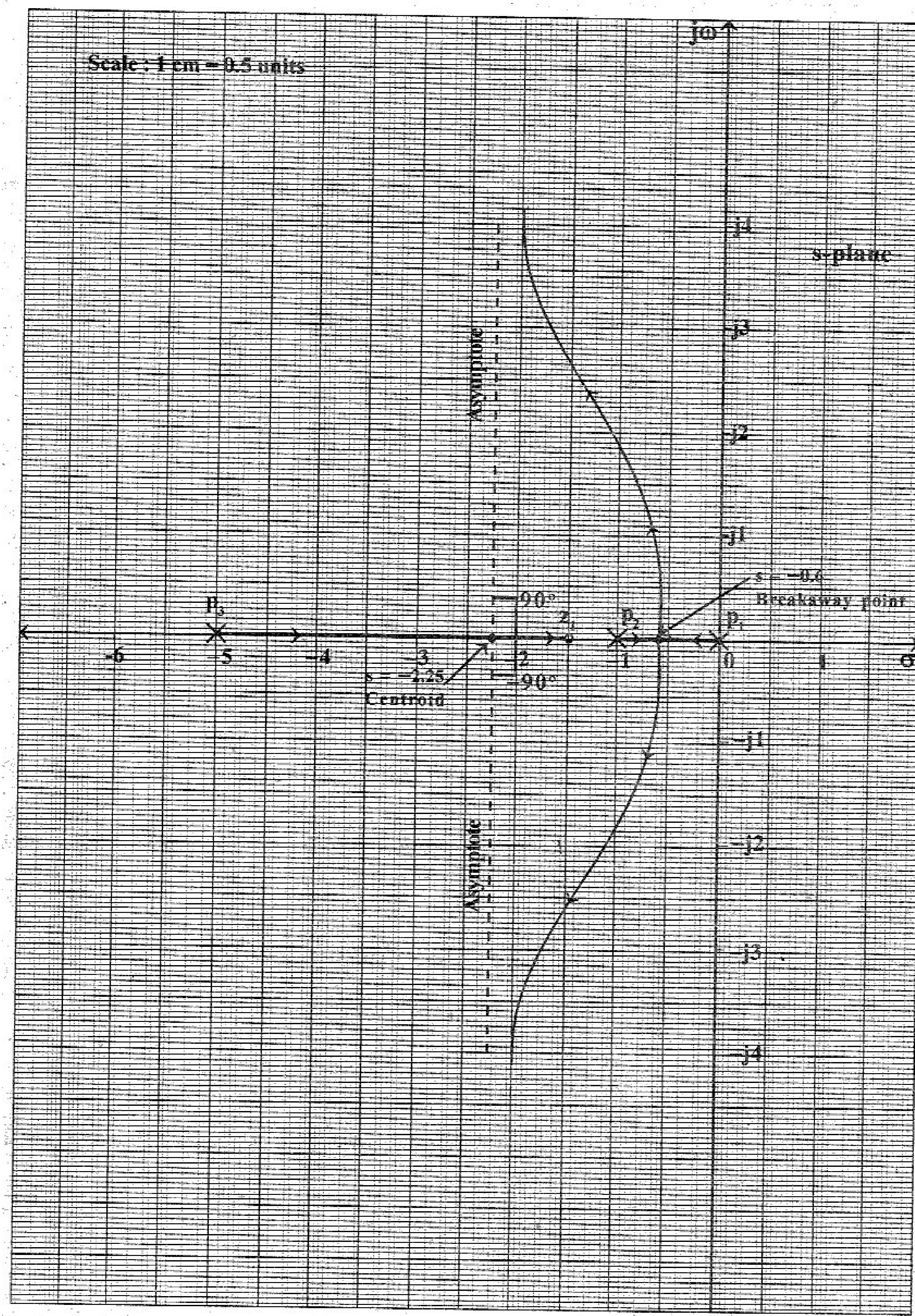


Fig 4.26.2. : Root locus sketch of, $1+G(s)=1+\dfrac{K(s+1.5)}{s(s+1)(s+5)}$.

The roots of the quadratic, $(s^2 + 4.65s + 6.2)$, are,

$$s = \frac{-4.65 \pm \sqrt{4.65^2 - 4 \times 6.2}}{2} = -2.3 \pm j0.89$$

Check for K : When $s = -0.6$, $K = \frac{-(s^3 + 6s^2 + 5s)}{s + 1.5} = \frac{-\left[(-0.6)^3 + 6(-0.6)^2 + 5(-0.6)\right]}{-0.6 + 1.5} = 1.17$

For $s = -0.6$, the value of K is positive and real and so it is actual breakaway point. It can be shown that for $s = -2.3 \pm j0.86$ the value of K is not positive and real and so they cannot be breakaway points. The actual breakaway point is shown in fig 4.26.1.

Step 5 : To find angle of departure

Since there are no complex pole or zero we need not find angle of departure or arrival.

Step 6 : To find crossing point of imaginary axis.

The characteristic equation is,

$$s\,(s+1)\,(s+5) + K\,(s+1.5) = 0 \;\Rightarrow\; s\,(s^2 + 6s + 5) + Ks + 1.5K = 0 \;\Rightarrow\; s^3 + 6s^2 + 5s + Ks + 1.5K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 5(j\omega) + K(j\omega) + 1.5K = 0 \;\Rightarrow\; -j\omega^3 - 6\omega^2 + j5\omega + jK\omega + 1.5K = 0$$

On equating imaginary part to zero, we get,

$$-j\omega^3 + j5\omega + jK\omega = 0$$
$$-j\omega^3 = -j5\omega - jK\omega$$
$$\omega^2 = 5 + K$$

On equating real part to zero we get,

$$-6\omega^2 + 1.5K = 0$$

put $\omega^2 = 5 + K$

$$-6(5+K) + 1.5K = 0$$
$$-30 - 4.5K = 0$$
$$-4.5K = 30 \;\Rightarrow\; K = -\frac{30}{4.5} = -6.67$$

Since the value of K is negative, there is no crossing point on imaginary axis, or for any positive values of K, and so the root locus will not cross imaginary axis.

The complete root locus sketch is shown in figure 4.26.2. The root locus has three branches. One branch starts at $s = -5$ and ends at finite zero at $s = -1.5$. The other two root locus starts at $s = 0$ and $s = -1$ and breakaway from real axis at $s = -0.6$, then travel parallel to asymptotes to meet the zeros at infinity.

EXAMPLE 4.27

Sketch root locus for the unity feedback system whose open loop transfer function is,

$$G(s) = \frac{K\,(s^2 + 6s + 25)}{s\,(s+1)\,(s+2)}.$$

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation $s(s+1)(s+2) = 0$ and the zeros are the roots of the equation $(s^2 + 6s + 25) = 0$.

The roots of quadratic are, $s = \frac{-6 \pm \sqrt{6^2 - 4 \times 25}}{2} = -3 \pm j4$

The poles are lying at, $s = 0, -1, -2$

The zeros are lying at, $s = -3 + j4, -3 - j4$.

Let us denote poles by p_1, p_2, p_3 and zeros by z_1, z_2.

Here, $p_1 = 0,\ p_2 = -1,\ p_3 = -2,\ z_1 = -3 + j4,\ z_2 = -3 - j4.$

The poles are marked by X (cross) and zeros by "o" (circle) as shown in fig 4.27.1.

Step 2 : To find root locus on real axis

The segment of real axis between s = 0 and s = -1 and the entire negative real axis from s = -2 will be part of root locus. Because if we choose a test point in this segment then to the right of this point we have odd number of real poles and zeros. The root locus on real axis are shown as a bold line in fig 4.27.1.

Step 3 : To find angles of asymptotes and centroid

Since there are three poles the number of root locus branches are three. There are two finite zeros, so two root locus branch will end at finite zeros. The third root locus will meet the zero at infinity by travelling through negative real axis. Here the number of asymptote is one and the angle of asymptote is ±180°.

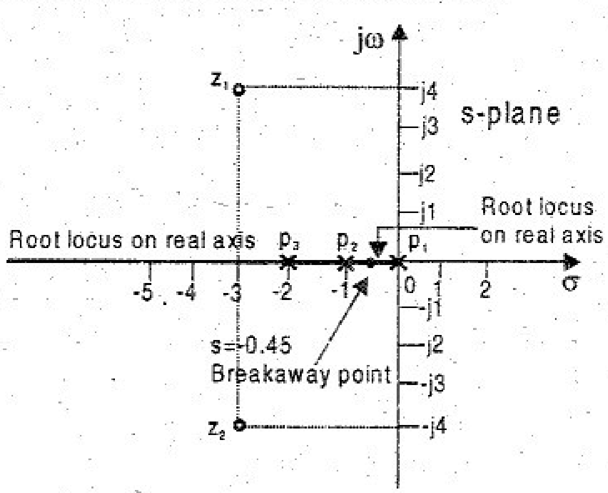


***Fig 4.27.1** : Figure showing the root locus on real axis location of poles, zeros and breakaway points.*

Step 4 : To find the breakaway and breakin points

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2)}}{1+\dfrac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2)}} = \frac{K\,(s^2+6s+25)}{s\,(s+1)\,(s+2)+K(s^2+6s+25)}$$

The characteristic equation is, $s\,(s+1)\,(s+2) + K\,(s^2+6s+25) = 0.$

$$\therefore K = \frac{-s\,(s+1)\,(s+2)}{s^2+6s+25} = \frac{-s\,(s^2+3s+2)}{s^2+6s+25} = \frac{-s^3-3s^2-2s}{s^2+6s+25}$$

On differentiating K with respect to s we get,

$$\frac{dK}{ds} = \frac{(-3s^2-6s-2)\,(s^2+6s+25)-(-s^3-3s^2-2s)\,(2s+6)}{(s^2+6s+25)^2}$$

$$= \frac{\begin{array}{c}-3s^4-18s^3-75s^2-6s^3-36s^2-150s-2s^2-12s-50\\ +2s^4+6s^3+6s^3+18s^2+4s^2+12s\end{array}}{(s^2+6s+25)^2} = \frac{-(s^4+12s^3+91s^2+150s+50)}{(s^2+6s+25)^2}$$

For $\dfrac{dK}{ds} = 0$, the numerator should be zero.

$$\therefore s^4+12s^3+91s^2+150s+50 = 0$$

The fourth order polynomial can be split into two quadratic equations. The two quadratic factors can be obtained by Lin's method. *(Refer Appendix-II).*

To find quadratic factors by Lin's Method.

The first trial divisor be the last three terms

Ist trial

$$\text{I}^{\text{st}}\ \text{Trial divisor} = 91s^2 + 150s + 50$$
$$= s^2 + \frac{150}{91}s + \frac{50}{91} = s^2 + 1.65s + 0.55$$

$$\begin{array}{r|l} & s^2 + 10.35s + 73.37 \\ \hline s^2 + 1.65s + 0.55 & s^4 + 12s^3 + 91s^2 + 150s + 50 \\ & s^4 + 1.65s^3 + 0.55s^2 \\ \hline & 10.35s^3 + 90.45s^2 + 150s \\ & 10.35s^3 + 17.08s^2 + 5.7s \\ \hline \text{II}^{\text{nd}}\ \text{trial divisor} \rightarrow & 73.37s^2 + 144.3s + 50 \\ & 73.37s^2 + 121.1s + 40.35 \\ \hline & 23.2s + 9.65 \end{array}$$

IInd trial

$$\text{II}^{\text{nd}}\ \text{Trial divisor} = 73.37s^2 + 144.3s + 50$$
$$= s^2 + \frac{144.3}{73.37}s + \frac{50}{73.37} = s^2 + 2s + 0.7$$

$$\begin{array}{r|l} & s^2 + 10s + 70.3 \\ \hline s^2 + 2s + 0.7 & s^4 + 12s^3 + 91s^2 + 150s + 50 \\ & s^4 + 2s^3 + 0.7s^2 \\ \hline & 10s^3 + 90.3s^2 + 150s \\ & 10s^3 + 20s^2 + 7s \\ \hline & 70.3s^2 + 143s + 50 \\ & 70.3s^2 + 140.6s + 49.2 \\ \hline & 2.4s + 0.8 \end{array}$$

On neglecting the small remainder we can write,

$$s^4 + 12s^3 + 91s^2 + 150s + 50 \approx (s^2 + 2s + 0.7)\ (s^2 + 10s + 70.3)$$

The roots of the quadratic, $s^2 + 2s + 0.7 = 0$, are,

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \times 0.7}}{2} = -0.45,\ -1.55$$

The roots of the quadratic, $s^2 + 10s + 70.3 = 0$, are,

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \times 70.3}}{2} = -5 \pm j6.73$$

Here, $s = -1.55$ is not a point on root locus, hence it cannot be a breakaway point.

Check the other three values for actual breakaway point.

$$\text{When } s = -0.45,\ K = \frac{-s^3 - 3s^2 - 2s}{s^2 + 6s + 25} = \frac{-(-0.45)^3 - 3(-0.45)^2 - 2(-0.45)}{(-0.45)^2 + 6(-0.45) + 25} = 0.017$$

For $s = -0.45$, the value of K is positive and real and so it is actual **breakaway point**. It can be shown that for $s = -5 \pm j6.73$ the value of K is not positive and real and so they cannot be breakaway points. The actual breakaway point is shown in fig 4.27.1.

Step 5 : To find angle of arrival

Let us consider the complex zero z_1 shown in fig 4.27.2. Draw vectors from all other poles and zero to the zero z_1 as shown in fig 4.27.2. Let the angles of these vectors be θ_1, θ_2, θ_3 and θ_4.

Here, $\theta_1 = 180° - \tan^{-1}\frac{4}{3} = 126.9°$

$\theta_2 = 180° - \tan^{-1}\frac{4}{2} = 116.6°$

$\theta_3 = 180° - \tan^{-1}\frac{4}{1} = 104°.$

$\theta_4 = 90°$

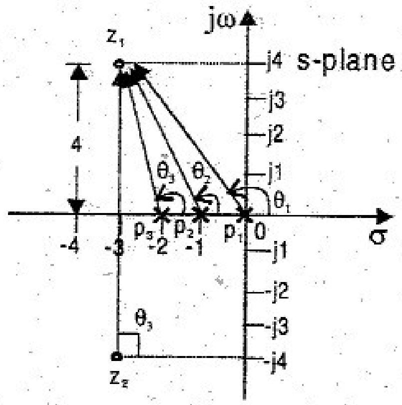


Fig 4.27.2

$$\left.\begin{array}{l}\text{Angle of arrival at}\\ \text{complex zero } z_1\end{array}\right\} = 180° - (\theta_4) + (\theta_1 + \theta_2 + \theta_3)$$

$$= 180° - 90° + 126.9° + 116.6° + 104°$$

$$= 437.5° = 77.5°$$

Angle of arrival at complex zero z_2 is negative of the angle of arrival at complex zero z_1.

$$\therefore \left.\begin{array}{l}\text{Angle of arrival at}\\ \text{complex zero } z_2\end{array}\right\} = -77.5°$$

Mark the angles of arrival at complex zeros using protractor.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is,

$$s(s+1)(s+2) + K(s^2+6s+25) = 0$$

$$s(s^2+3s+2) + Ks^2 + 6Ks + 25K = 0$$

$$s^3 + 3s^2 + 2s + Ks^2 + 6Ks + 25K = 0$$

$$s^3 + (3+K)s^2 + (2+6K)s + 25K = 0$$

Put $s = j\omega$.

$$(j\omega)^3 + (3+K)(j\omega)^2 + (2+6K)(j\omega) + 25K = 0 \Rightarrow -j\omega^3 - (3+K)\omega^2 + j(2+6K)\omega + 25K = 0$$

On equating imaginary part to zero

$$-j\omega^3 + j(2+6K)\omega = 0$$

$$-j\omega^3 = -j(2+6K)\omega$$

$$\omega^2 = (2+6K)$$

On equating real part to zero

$$-(3+K)\omega^2 + 25K = 0$$

Put $\omega^2 = 2 + 6K$

$$-(3+K)(2+6K) + 25K = 0$$

$$-(6+18K+2K+6K^2) + 25K = 0$$

$$-6K^2 + 5K - 6 = 0$$

$$K = \frac{-5 \pm \sqrt{5^2 - 4\times(-6)(-6)}}{2\times(-6)} = 0.4 \pm j0.9$$

Since the value of K is not real and positive, there is no crossing point on imaginary axis, or for any positive values of K the root locus will not cross imaginary axis.

Step 7 : To find points on root locus

Choose test points a, b, c, d on the s-plane and adjust the test points to satisfy angle criterion. The test points are shown in fig 4.27.3.On the upper half of s-plane the root locus is sketched through the test points a, b, c and d. The root locus on the lower half of s-plane is the mirror image of the root locus on the upper half of s-plane.

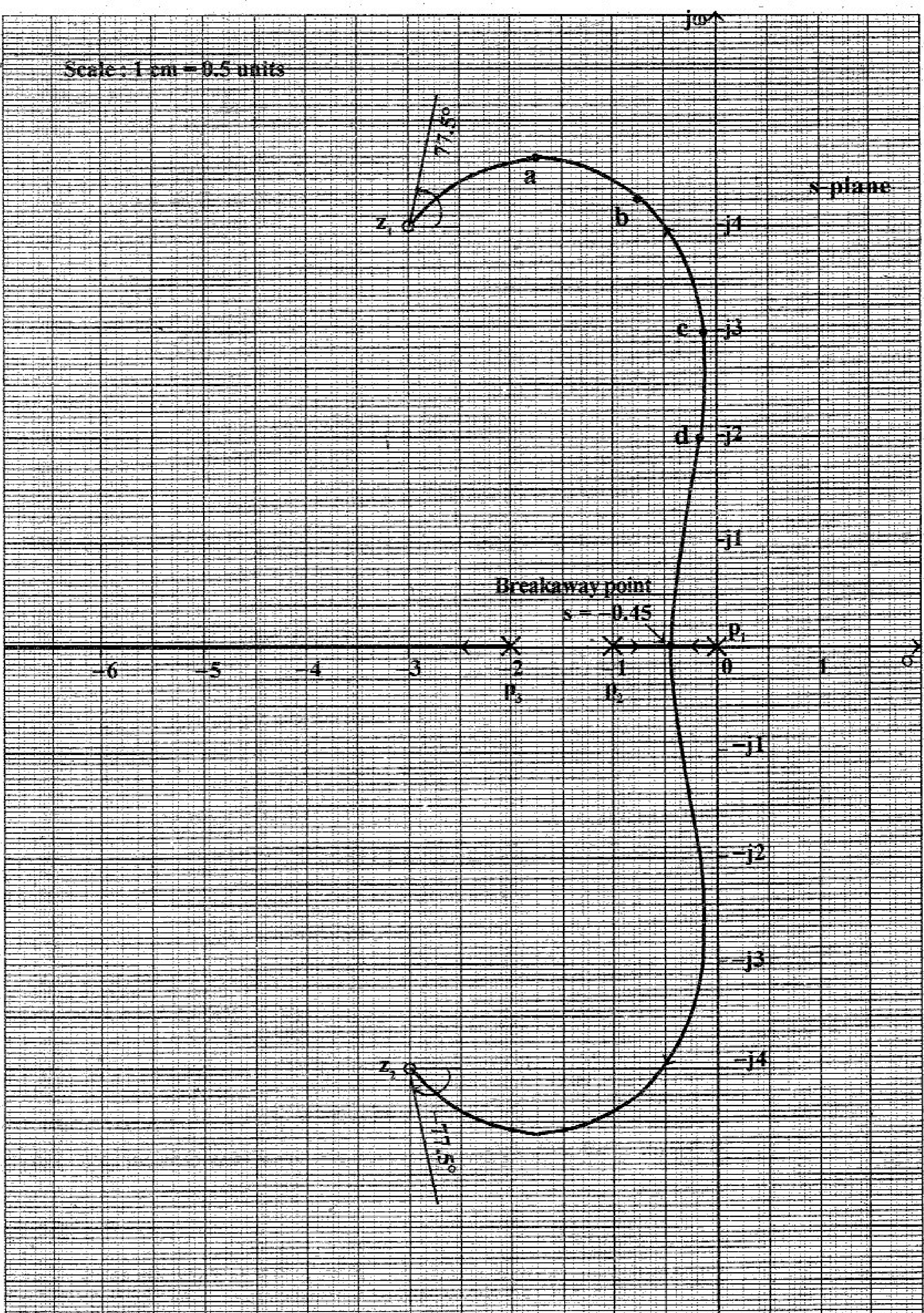


Fig 4.27.3. : Root locus sketch of, $1+G(s)=1+\dfrac{K(s^2+6s+25)}{s\,(s+1)\,(s+2)}$.

The complete root locus sketch is shown in fig 4.27.3. The root locus has three branches. One branch starts at s = -2 and goes to infinity along negative real axis. The other two root locus branches starts at s = 0 and s = -1 and breaks from real axis at s = -0.45, then meets the complex zeros.

EXAMPLE 4.28

Sketch the root locus for the unity feedback system whose open loop transfer function is,

$$G(s) = \frac{K}{s(s^2+6s+10)}.$$

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s^2+6s+10)=0$.

The roots of the quadratic are, $s = \frac{-6 \pm \sqrt{6^2 - 4 \times 10}}{2} = -3 \pm j1$

The poles are lying at, s= 0, –3+j1 and –3 –j1

Let us denote the poles as p_1, p_2, and p_3.

Here, $p_1 = 0$, $p_2 = -3+j1$, and $p_3 = -3-j1$.

The poles are marked by X(cross) as shown in fig 4.28.1

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown are three.

Note : For the given transfer function one root locus branch will start at the pole at the origin and meet the zero at infinity through the negative real axis.

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no infinite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}; \qquad q = 0,\ 1,\ 2,\ ....n-m.$$

Here, n = 3 and m = 0. $\therefore$ q = 0, 1, 2, 3,.....

When q = 0, $\quad \text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$

When q = 1, $\quad \text{Angles} = \pm\frac{180^\circ \times 3}{3} = \pm 180^\circ$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m} = \frac{-3+j1-3-j1}{3} = -2$$

The centroid is marked on real axis and from the centroid the angles of asymptotes are marked using a protractor. The asymptotes are drawn as dotted lines as shown in fig 4.28.1.

Step 4: To find the breakaway and breakin poir.ts

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s^2+6s+10)}}{1+\frac{K}{s(s^2+6s+10)}} = \frac{K}{s(s^2+6s+10)+K}$

The characteristic equation is, $s(s^2+6s+10)+K=0$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -3s^2 - 12s - 10$$

Put $\frac{dK}{ds} = 0$

$$-3s^2 - 12s - 10 = 0 \quad \Rightarrow \quad 3s^2 + 12s + 10 = 0$$

$$\therefore s = \frac{-12 \pm \sqrt{12^2 - 4 \times 3 \times 10}}{2 \times 3} = -1.18 \text{ or } -2.82$$

Check for K : When, $s = -1.18$, $K = -s^3 - 6s^2 - 10s = -(-1.18)^3 - 6(-1.18)^2 - 10(-1.18) = 5.09$

When, $s = -2.82$, $K = -s^3 - 6s^2 - 10s = -(-2.82)^3 - 6(-2.82)^2 - 10(-2.82) = 2.91$

- Since the values of K for $s = -1.18$ and -2.82 are positive and real, both the points are actual breakaway or breakin points. It can be proved that $s = -2.82$ is a breakin point and $s = -1.18$ is a breakaway point. The breakin and breakaway points are shown in fig 4.28.1.

[Also the value of K for $s = -2.82$ is less than the value of K for $s = -1.18$, therefore when root locus travel from $s = -2.82$ to -1.18, the value of K increases]

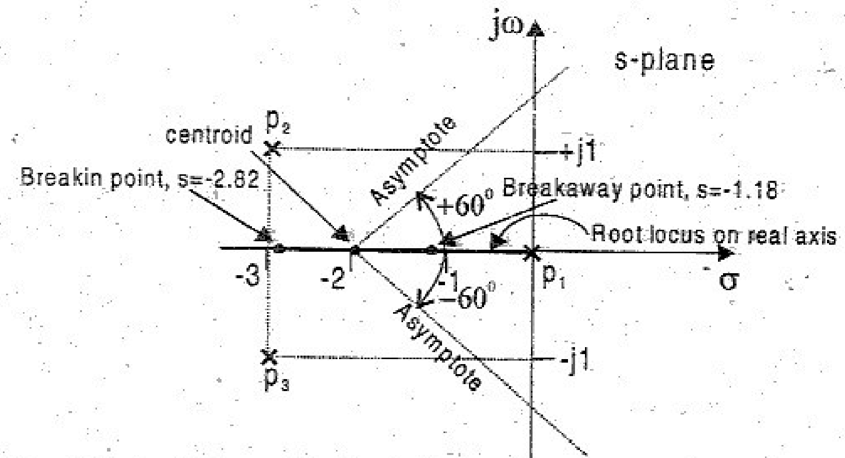


Fig 4.28.1 : Figure showing the asymptotes, root locus on real axis and location of poles, zeros, centroid, breakin and breakaway points.

Step 5: To find the angle of departure

Consider the complex pole p_2 shown in fig 4.28.2. Draw vectors from all other poles to the pole p_2 as shown in fig 4.28.2. Let the angle of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(1/3) = 161.6°$

$\theta_2 = 90°$

$$\left.\begin{array}{l}\text{Angle of departure from} \\ \text{the complex pole } p_2\end{array}\right\} = 180° - (\theta_1 + \theta_2)$$

$$= 180° - (161.6° + 90°) = -71.6° \approx -72°$$

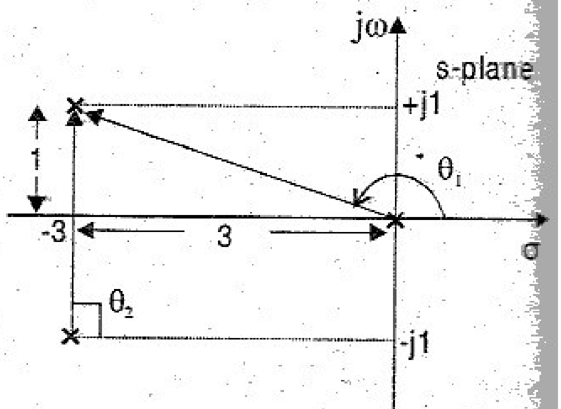


Fig 4.28.2

The angle of departure at complex pole p_3 is negative of the angle of departure at complex pole p_2.

$\therefore$ Angle of departure at pole $p_3 = +72$

Mark the angles of departure at complex poles using protractor.

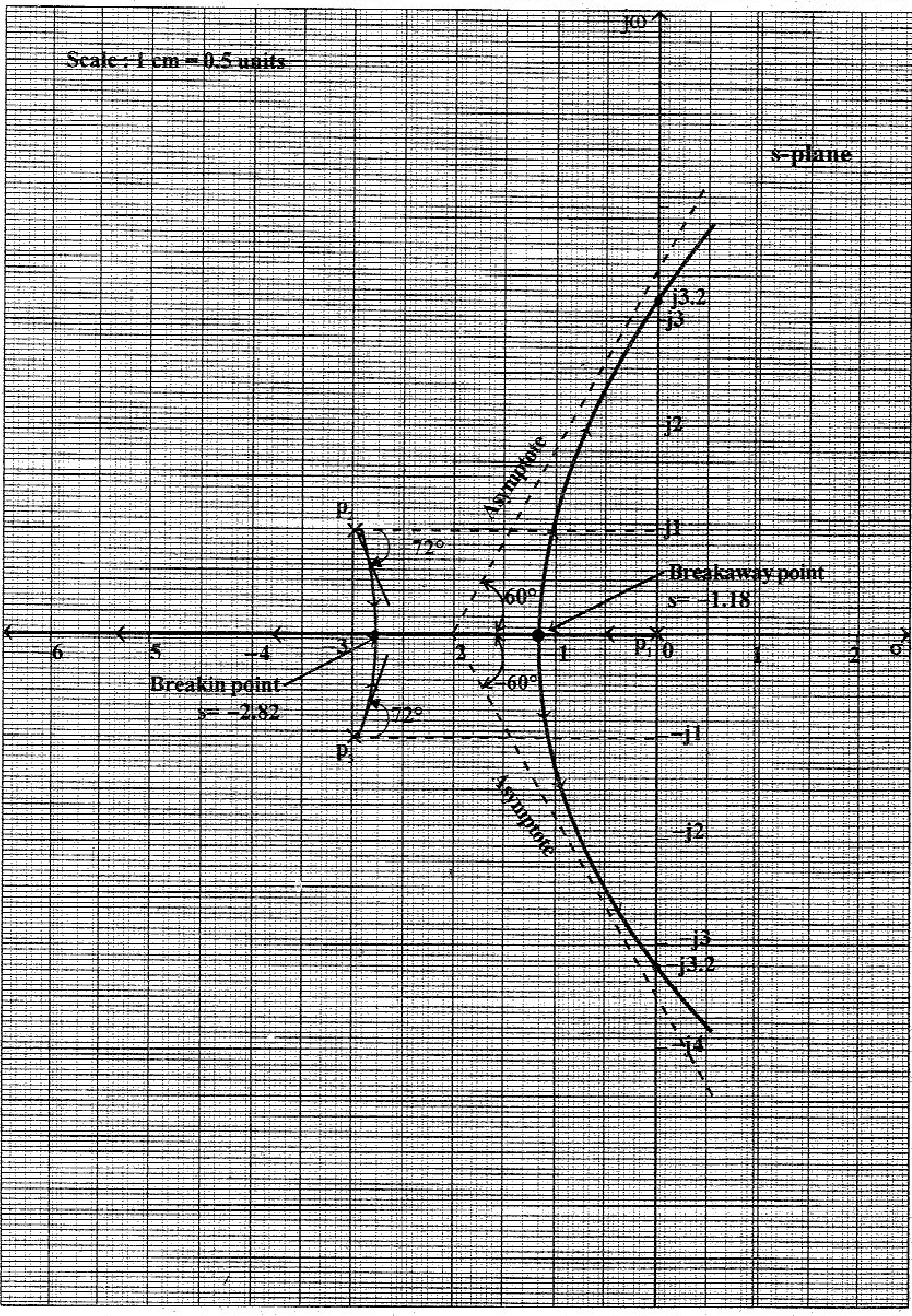


Fig 4.28.3. : Root locus sketch of, $1+G(s)=1+\dfrac{K}{s\,(s^2+6s+10)}$.

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by, $s(s^2+6s+10)+K=0 \Rightarrow s^3+6s^2+10s+K=0$

Put $s = j\omega$.

$$(j\omega)^3+6(j\omega)^2+10(j\omega)+K=0 \Rightarrow -j\omega^3-6\omega^2+j10\omega+K=0$$

On equating imaginary part to zero we get,

$$-\omega^3+10\omega=0$$
$$\omega^3=10\omega$$
$$\omega^2=10$$
$$\omega=\pm\sqrt{10}=\pm 3.16\approx\pm 3.2$$

On equating real part to zero we get,

$$-6\omega^2+K=0$$
$$K=6\omega^2$$
$$=6\times 10=60$$

The root locus crosses imaginary axis at $\pm j3.2$ and the gain K corresponding to this point is 60. This is the limiting value of K for the stability of the system.

The complete root locus sketch is shown in fig 4.28.3. The root locus has three branches. One branch starts at $s = 0$ and goes to infinity along negative real axis. The other two root locus branches starts at $s = -3 \pm j1$ and enter the real axis at $s = -2.82$ and then breakaway from real axis at $s = -1.18$. Finally they travel parallel to asymptotes to meet the zeros at infinity.

4.9 NYQUIST AND ROOT LOCUS PLOTS USING MATLAB

In general, the open loop transfer function of a system is denoted as G(s).

Let, G(s) be a rational function of "s", as shown below.

$$G(s)=\frac{b_0 s^M+b_1 s^{M-1}+b_2 s^{M-2}+\ldots\ldots\ldots+b_{M-1}s+b_M}{a_0 s^N+a_1 s^{N-1}+a_2 s^{N-2}+\ldots\ldots\ldots+a_{N-1}s+a_N}$$

For drawing Nyquist and root locus plots, the transfer function G(s) is declared as a function of s using the following commands.

```
s=tf('s');
Gs=(b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
```

The coefficients of numerator and denominator polynomials of the transfer function are determined using the following command.

```
[num_cof den_cof]=tfdata(Gs);
```

The horizontal and vertical axes range for the Nyquist and root locus plots can be specified using the axis command as shown below.

```
axis([x_start  x_end  y_start  y_end]);
```

NYQUIST PLOT

The Nyquist plot can be plotted using any one of the following commands.

```
nyquist(Gs);
nyquist(Gs,'k');
nyquist(num_cof, den_cof);
```

ROOT LOCUS PLOT

The root locus plot can be plotted using any one of the following commands.

```
rlocus(Gs);
rlocus(Gs,'k');
rlocus(num_cof, den_cof);
```

PROGRAM 4.1

Write a MATLAB program to draw the Nyquist plot of the system governed by the following open loop transfer function.

G(s) = 240/s(s+2)(s+10).

```
%program to plot Nyquist plot

clear all
clc
s=tf('s');
disp('The given transfer function is');
Gs=240/(s*(s+2)*(s+10))

nyquist(Gs,'k');
axis([-4 0.5 -2 2]); grid;
```

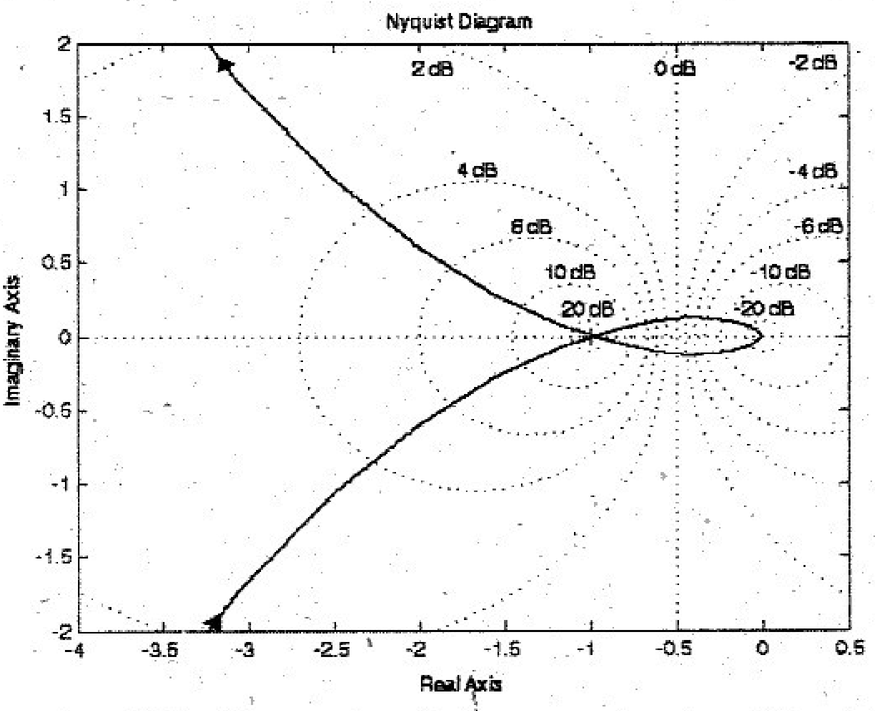


Fig P4.1 : Nyquist plot of the system given in problem 4.1.

OUTPUT

The given transfer function is,

Transfer function:

```
                     240
             --------------------
             s^3 + 12 s^2 + 20 s
```

The Nyquist plot of program 4.1 is shown in fig P4.1.

PROGRAM 4.2

Write a MATLAB program to draw the Nyquist plot of the system governed by the following open loop transfer function.

G(s) = (1+0.5s)(1+s)/(1+10s)(s-1)

```
%program to plot Nyquist plot

clear all
clc
```

```
s=tf('s');
disp('The given transfer function is,')
Gs=((1+0.5*s)*(1+s))/((1+10*s)*(s-1))

nyquist(Gs,'k');
axis([-1.2 0.2 -1 1]);
grid;
```

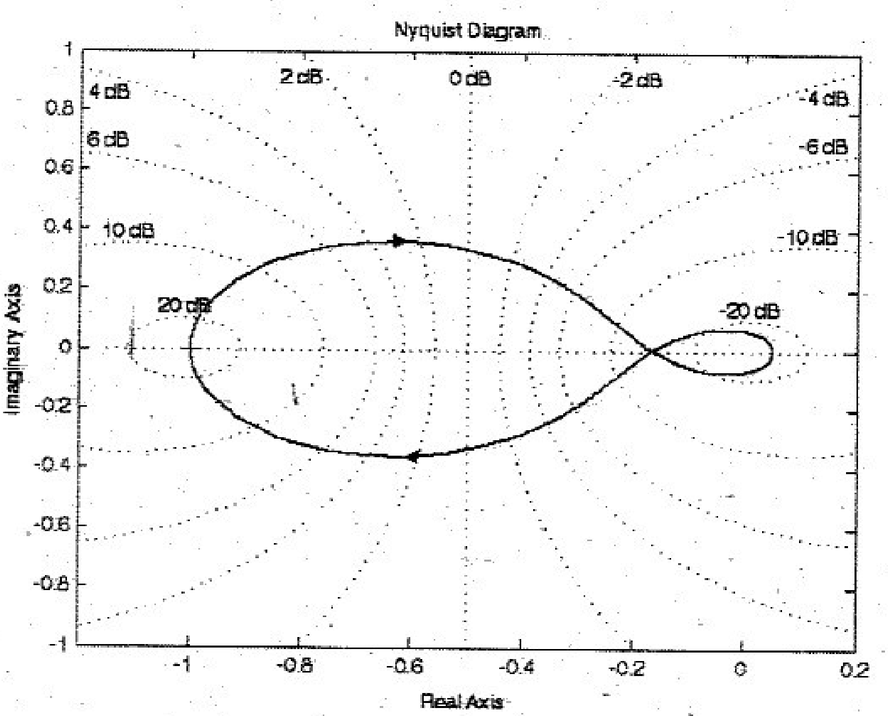


Fig P4.2 : Nyquist plot of the system given in problem 4.2.

OUTPUT

```
The given transfer function is,

Transfer function:
                    0.5 s^2 + 1.5 s + 1
                    -------------------
                    10 s^2 - 9 s - 1

The Nyquist plot of program 4.2 is shown in fig P4.2.
```

PROGRAM 4.3

Write a MATLAB program to draw the Nyquist plot of the system governed by the following open loop transfer function.

G(s) = (s+2)/(s+1)(s-1)

```
%program to plot Nyquist plot

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=(s+2)/((s+1)*(s-1))

nyquist(Gs,'k');
axis([-2.5 0.2 -1 1]);
grid;
```

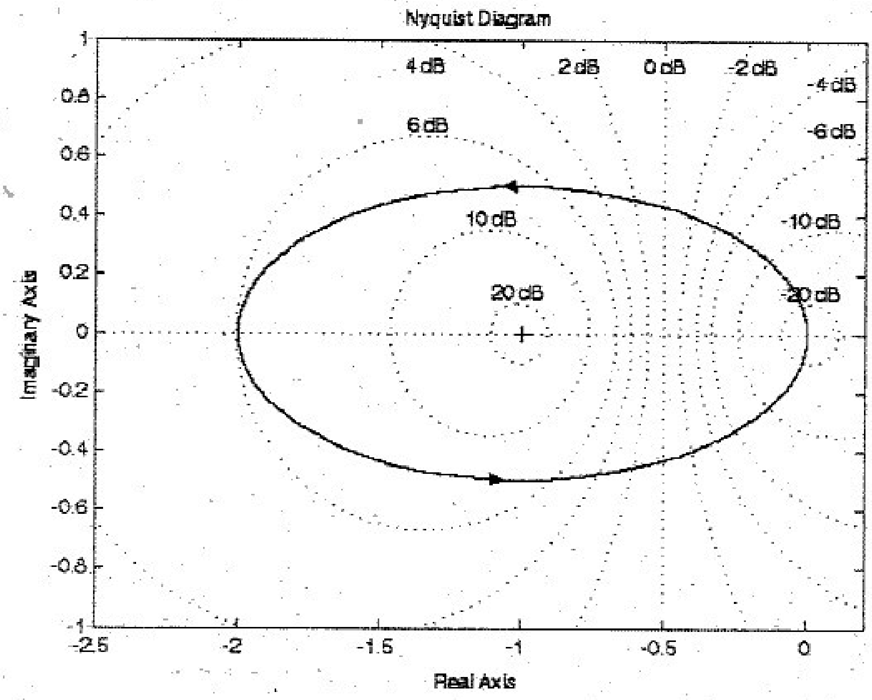


Fig P4.3 : Nyquist plot of the system given in problem 4.3.

OUTPUT

```
The given transfer function is,
Transfer function:
                    s + 2
                    ------
                    s^2 - 1

The Nyquist plot of program 4.3 is shown in fig P4.3.
```

PROGRAM 4.4

Write a MATLAB program to draw the root locus plot of the unity feedback system governed by the following open loop transfer function.

$$G(s) = 1/s(s^2+4s+13)$$

```
%program to plot root locus

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=1/(s*(s^2+4*s+13))
rlocus(Gs,'k');
axis([-3 2 -6 6]);                    %specify x and y axis limits
sgrid([0.5,0.707],[90.5,1,2]);   %specify the s-grid lines to draw
```

OUTPUT

```
The given open loop transfer function G(S) is,
Transfer function:
                        1
                -----------------
                s^3 + 4 s^2 + 13 s

The root locus plot of program 4.4 is shown in fig P4.4.
```

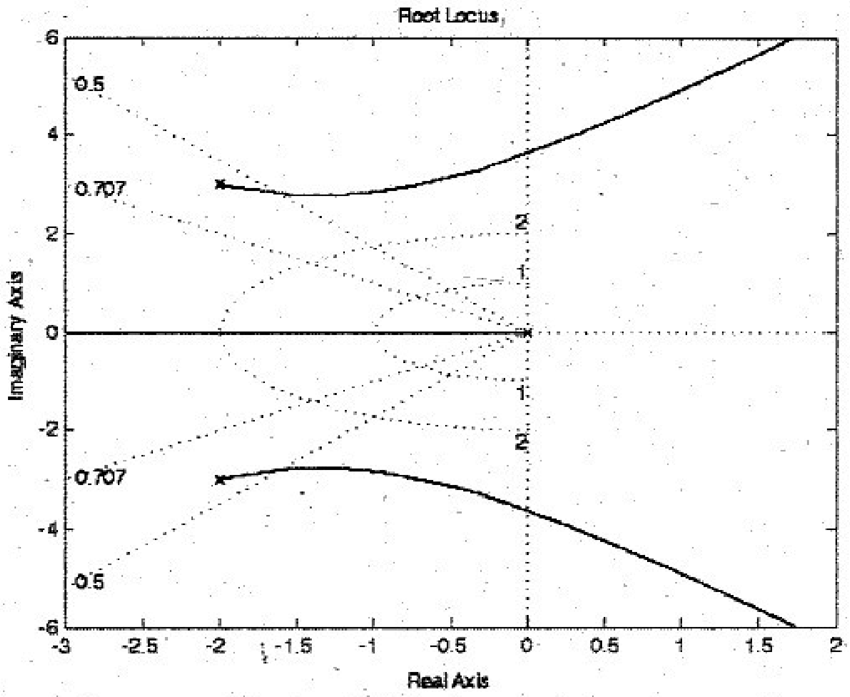


Fig P4.4 : Root locus plot of the system given in problem 4.4.

PROGRAM 4.5

Write a MATLAB program to draw the root locus plot of the unity feedback system governed by the following open loop transfer function.

$$G(s) = 1/s(s+4)(s^2+4s+20)$$

```
%program to plot root locus
clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=1/(s*(s+4)*(s^2+4*s+20))
rlocus(Gs,'k'); axis([-8 4 -6 6]); sgrid;
```

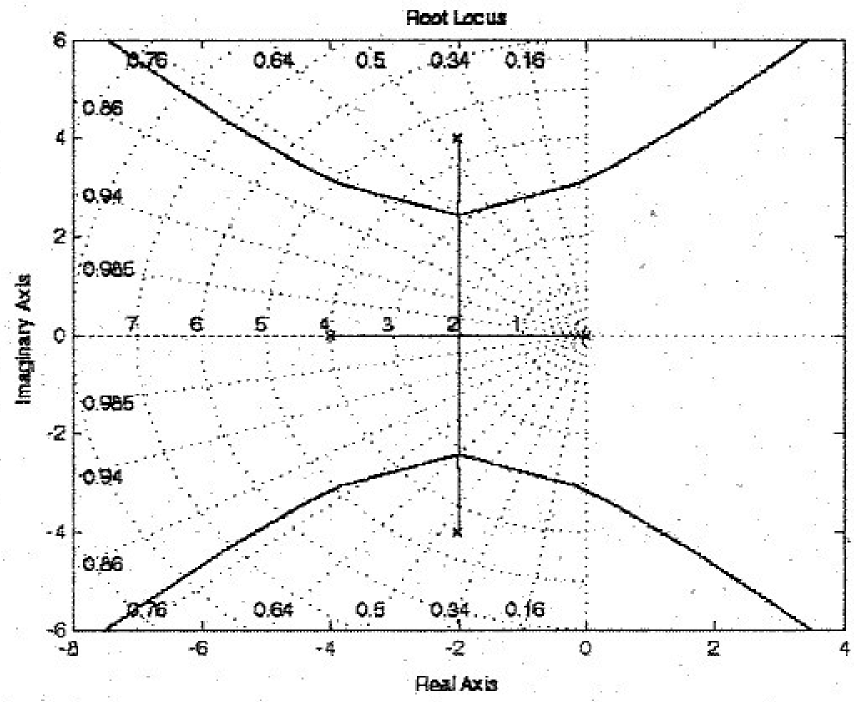


Fig P4.5 : Root locus plot of the system given in problem 4.5.

OUTPUT

```
The given open loop transfer function G(S) is,
Transfer function:
                        1
          ----------------------------
          s^4 + 8 s^3 + 36 s^2 + 80 s
```

The root locus plot of program 4.5 is shown in fig P4.5.

PROGRAM 4.6

Write a MATLAB program to draw the root locus plot of the unity feedback system governed by the following open loop transfer function.

$G(s) = (s^2+6s+25)/s(s+1)(s+2)$

```
%program to plot root locus

clear all
clc
s=tf('s');
disp('The given transfer function is,');
Gs=(s^2+6*s+25)/(s*(s+1)*(s+2))

rlocus(Gs,'k');
axis([-6 2 -6 6]); sgrid;
```

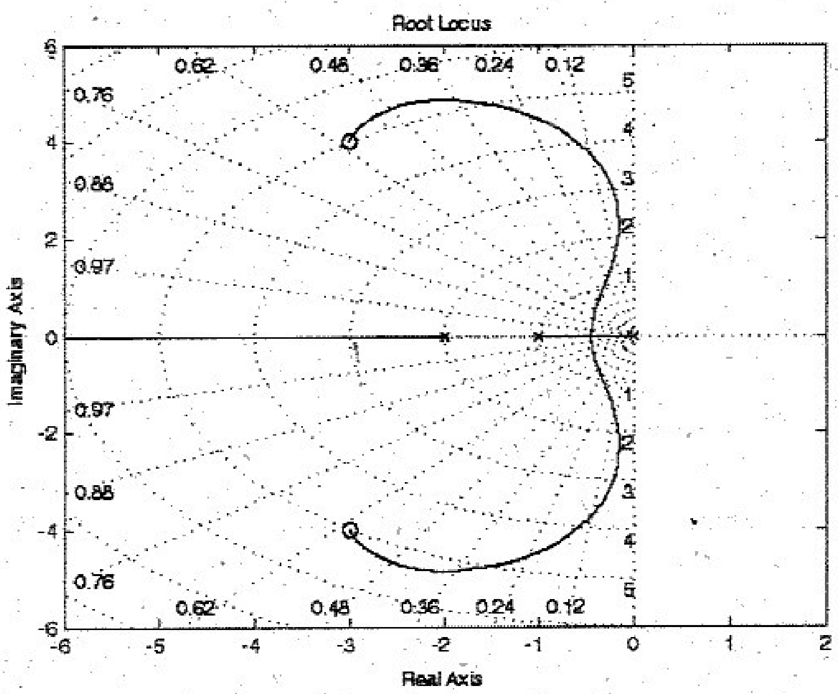


Fig P4.6 : Root locus plot of the system given in problem 4.6.

OUTPUT

```
The given open loop transfer function G(S) is,
Transfer function:
             s^2 + 6 s + 25
            -----------------
             s^3 + 3 s^2 + 2 s
```

The root locus plot of program 4.6 is shown in fig P4.6.

4.10 SHORT QUESTIONS AND ANSWERS

Q4.1 ***Define BIBO stability.***

A linear relaxed system is said to have BIBO stability if every bounded (finite) input results in a bounded (finite) output.

Q4.2 ***What is impulse response?***

The impulse response of a system is the response of a system for impulse input and it is given by inverse Laplace transform of the system transfer function.

Q4.3 ***What is the requirement for BIBO stability?***

The requirement for BIBO stability is that, $\int_0^{\infty} m(t)\, dt < \infty$,

where m(t) is impulse response of the system.

Q4.4 ***What is characteristic equation?***

The denominator polynomial of C(s)/R(s) is the characteristic equation of the system.

Q4.5 ***How the roots of characteristic equation are related to stability?***

If the roots of characteristic equation has positive real part then the impulse response of the system is not bounded (the impulse response will be infinite as $t \to \infty$). Hence the system will be unstable. If the roots have negative real part then the impulse response is bounded (the impulse response becomes 0 as $t \to \infty$). Hence the system will be stable.

Q4.6 ***What is the necessary condition for stability?***

The necessary condition for stability is that all the coefficients of the characteristic polynomial must be positive.

Q4.7 ***What is the relation between stability and coefficient of characteristic polynomial?***

If the coefficients of characteristic polynomial are negative or zero, then some of roots lie on right half of s-plane. Hence the system is unstable. If the coefficients of characteristic polynomial are positive and if no coefficient is zero then there is a possibility of the system to be stable provided all the roots are lying on left half of s-plane.

Q4.8 ***What will be the nature of impulse response when the roots of characteristic equation are lying on imaginary axis?***

If the roots of characteristic equation lies on imaginary axis the nature of impulse response is oscillatory.

Q4.9 ***What will be the nature of impulse response if the roots of characteristic equation are lying on right half of s-plane?***

When the roots are lying on the real axis on the right half of s-plane, then the response is exponentially increasing. When the roots are complex conjugate and lying on the right half of s-plane, then the response is oscillatory with exponentially increasing amplitude.

Q4.10 ***What is the principle of argument?***

The principle of argument states that let F(s) be an analytic function and if an arbitrary closed contour in the clockwise direction is chosen in the s-plane so that F(s) is analytic at every point of the contour. Then the corresponding F(s)-plane contour mapped in the F(s)-plane will encircle the origin, N times in the anticlockwise direction, where N is the difference between number of poles ,P and zeros Z of F(s) that are enclosed by the chosen closed contour in the s-plane. (i.e., N = P -Z).

Q4.11 ***What is the necessary and sufficient condition for stability?***

The necessary and sufficient condition for stability is that all of the elements in the first column of the routh array should be positive.

Q4.12 ***What is routh stability criterion?***

Routh criterion states that the necessary and sufficient condition for stability is that all of the elements in the first column of the routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of routh array corresponds to the number of roots of characteristic equation in the right half of the s-plane.

Q4.13 ***What is auxiliary polynomial?***

In the construction of routh array a row of all zero indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is called auxiliary polynomial. The coefficients of auxiliary polynomial are given by the elements of the row just above the row of all zeros.

Q4.14 ***What is quadrantal symmetry?***

The symmetry of roots with respect to both real and imaginary axis is called quadrantal symmetry.

Q4.15 ***In routh array what conclusion you can make when there is a row of all zeros?***

All zero row in routh array indicates the existence of an even polynomial as a factor of the given characteristic equation. The even polynomial may have roots on imaginary axis.

Q4.16 ***What is limitedly stable system ?***

For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable.

Q4.17 ***What is Nyquist stability criterion?***

If G(s)H(s)-contour in the G(s)H(s)-plane corresponding to Nyquist contour in s-plane encircles the point $-1+j0$ in the anti-clockwise direction as many times as the number of right half s-plane poles of G(s)H(s). Then the closed loop system is stable.

Q4.18 ***What is root locus?***

The path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus.

Q4.19 ***What is magnitude criterion?***

The magnitude condition states that $s=s_a$ will be a point on root locus if for that value of s magnitude of G(s)H(s) is equal to 1, (i.e. $|G(s)H(s)| = 1$).

$$\text{Let, } G(s)\,H(s) = \frac{K\,(s+z_1)\,(s+z_2)\,(s+z_3)\,.....}{(s+p_1)\,(s+p_2)\,(s+p_3)\,.....}$$

$\therefore$ For $s = s_a$ be a point in root locus,

$$|G(s)\,H(s)| = \frac{K\,|s_a+z_1|\,|s_a+z_2|\,|s_a+z_3|\,.....}{|s_a+p_1|\,|s_a+p_2|\,|s_a+p_3|\,.....} = 1$$

$$\left[\text{or } |G(s)\,H(s)| = K = \frac{\text{Product of length of vectors from open loop zeros to the point } s_a}{\text{Product of length of vectors from open loop poles to the point } s_a} = 1\right]$$

Q4.20 ***What is angle criterion?***

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s the argument or phase of G(s)H(s) is equal to an odd multiple of 180°, [i.e., $\angle G(s)H(s) = \pm 180° \,(2q+1)$].

$$\text{Let, } G(s)H(s) = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,.....}{(s+p_1)\,(s+p_2)\,(s+p_3)\,.....}$$

$\therefore$ For $s = s_a$ be a point on root locus,

$$\angle G(s)\,H(s) = \angle(s_a+z_1) + \angle(s_a+z_2) + \angle(s_a+z_3)\,.....-\angle(s_a+p_1) - \angle(s_a+p_2)\,.... = \pm 180°(2q+1)$$

$$\left[\text{or} \left(\begin{array}{c} \text{Sum of anges} \\ \text{of vectors from zeros} \\ \text{to the point } s = s_a \end{array} \right) - \left(\begin{array}{c} \text{Sum of angles} \\ \text{of vectors from poles} \\ \text{to the point } s = s_a \end{array} \right) = \pm 180°(2q+1) \right]$$

Q4.21. ***How will you find the gain K at a point on root locus?***

The gain K at a point $s = s_a$ on root locus is given by,

$$K = \frac{\text{Product of length of vector from open loop poles to the point } s_a}{\text{Product of length of vector from open loop zeros to the point } s_a}$$

Q4.22 ***How will you find root locus on real axis?***

To find the root locus on real axis, choose a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Q4.23 ***What are asymptotes? How will you find the angle of asymptotes?***

Asymptotes are straight lines which are parallel to root locus going to infinity and meet the root locus at infinity.

$$\text{Angles of asymptotes} = \frac{\pm 180° \ (2q+1)}{n-m} \ ; \ q = 0, 1, 2, \ldots\ldots (n-m)$$

Q4.24 ***What is centroid? How the centroid is calculated?***

The meeting point of asymptotes with real axis is called centroid. The centroid is given by,

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Q4.25 ***What are breakaway and breakin point? How to determine them?***

At breakaway point the root locus breaks from the real axis to enter into the complex plane. At breakin point the root locus enters the real axis from the complex plane.

To find the breakaway or breakin points, form an equation for K from the characteristic equation, and differentiate the equation of K with respect to s. Then find the roots of equation $dK/ds = 0$. The roots of $dK/ds = 0$ are breakaway or breakin points, provided for this value of root, the gain K should be positive and real.

Q4.26 ***How to find the crossing points of root locus in imaginary axis.***

Method (i) : By Routh hurwitz criterion.

Method (ii) : By letting $s = j\omega$ in the characteristic equation and separate the real and imaginary parts. These two equations are equated to zero. Solve the two equations for ω and K. The value of ω gives the point where the root locus crosses imaginary axis and the value of K is the gain corresponding to the crossing point.

Q4.27 ***What is dominant pole?***

The dominant pole is a pair of complex conjugate pole which decides transient response of the system. In higher order systems the dominant poles are very close to origin and all other poles of the system are widely separated and so they have less effect on transient response of the system.

Q4.28 ***How will you fix dominant pole on root locus and find the gain K corresponding to the dominant pole?***

The dominant poles are given by roots of a quadratic factor, $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$.

$$\therefore s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

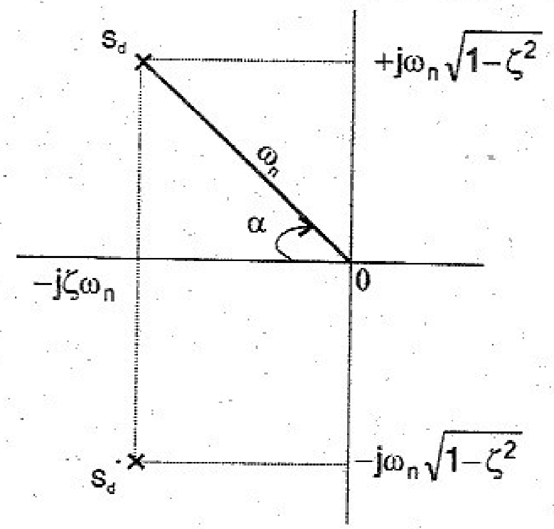


Fig Q4.28

The dominant pole can be plotted on the s-plane as shown in fig Q4.28.

In the right angle triangle OAP,

$$\cos\alpha = \frac{\zeta\omega_n}{\omega_n} = \zeta \qquad \therefore \alpha = \cos^{-1}\zeta$$

To fix a dominant pole on root locus draw a line at an angle of $\cos^{-1}\zeta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

$$K = \frac{\text{Product of length of vectors from open loop poles to dominant pole}}{\text{Product of length of vectors from open loop zeros to dominant pole}}$$

Q4.29 ***For the system represented by the following characteristic equation say whether the necessary condition for stability is satisfied or not.***

(i) $s^4 + 3s^3 + 4s^2 + 5s + 10 = 0$

(ii) $s^6 - 2s^5 + s^3 + s^2 + s + 6 = 0$

(iii) $s^3 - 8s^2 + 7s + 6 = 0$

(iv) $s^5 + 4s^4 - 5s^3 - 4s^2 + 2s + 1 = 0$

In equation (i) All the coefficients are positive and so the necessary condition for stability is satisfied.

In equation (ii), (iii) and (iv) some of the coefficients are negative and some of the coefficients are missing. Hence the necessary condition for stability is not satisfied.

Q4.30 ***Check the stability of the system whose Nyquist plot are shown in figQ 4.30 a & b, if there is no open loop poles on right half of s-plane.***

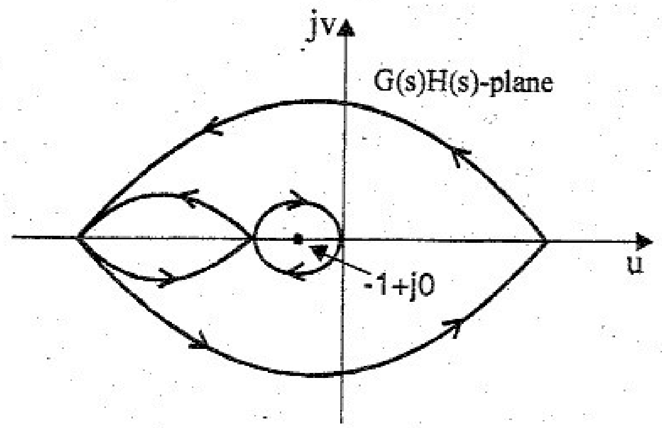


Fig Q4.30 a

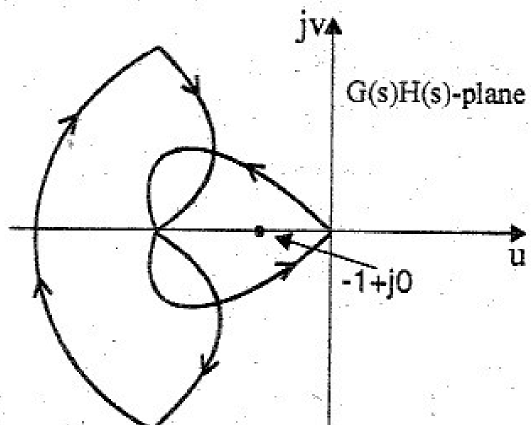


Fig Q4.30 b

In contour shown in fig Q 4.30a the $-1 + j0$ point is encircled once in clockwise direction and once in anticlockwise direction. Hence net encirclement is zero. Since no poles are lying on right half of s-plane and net encirclement of $-1+j0$ is zero, and so the system is stable.

In contour shown in fig Q4.30b the $-1+j0$ point is encircled once in anticlockwise direction but there is no pole on right half hence and hence the system is unstable.

4.11 EXERCISES

E.4.1 *Using routh criterion determine the locations of the roots of the following characteristic equations and comment on the stability of the systems.*

a) $2s^5 + 2s^4 + 5s^3 + 5s^2 + 3s + 5 = 0$

b) $3s^4 + 10s^3 + 5s^2 + 5s + 3 = 0$

c) $2s^6 + 4s^5 + s^4 - 32s^3 + 51s^2 + 3s + 15 = 0$

d) $s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$

e) $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

f) $s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$

E.4.2 *The characteristic equations for certain feedback control systems are given below. In each case, determine the range of values of K, for which the system is stable.*

a) $s^4 + 3s^3 + 3s^2 + s + K = 0$

b) $s^5 + s^4 + Ks^3 + s^2 + s + 1 = 0$

c) $s^5 + s^4 + s^2 + s + K = 0$

d) $s^4 + s^3 + 3Ks^2 + (K + 2)s + 4 = 0$

e) $s^4 + s^3 + 3(K + 1)s^2 + (7K + 5)s + (4K + 7) = 0$

E.4.3 *Open-loop transfer functions of certain unity feedback systems are given below. In each case determine the location of closed loop poles in the s-plane, using routh criterion. Comment on the stability of closed loop system.*

a) $G(s) = \dfrac{200\,(1+s)}{s\,(1+0.1s)\,(1+0.2s)\,(1+0.5s)}$

b) $G(s) = \dfrac{10}{(s+2)\,(s+4)\,(s^2+6s+25)}$

c) $G(s) = \dfrac{(s+1)}{s(s-1)\,(s^2+4s+16)}$

d) $G(s) = \dfrac{2.5}{s(s+5)\,(0.1s+1)}$

E.4.4 *Open-loop transfer functions of certain unity feedback systems are given below. In each case determine the range of values of K for which the system is stable.*

a) $G(s) = \dfrac{K(s+13)}{s(s+3)(s+7)}$

b) $G(s) = \dfrac{K(s+1)}{s(s-1)(s+6)}$

c) $G(s) = \dfrac{K(s+2)}{s(s+5)(s^2+2s+5)}$

d) $G(s) = \dfrac{K(s-1)}{s(s+2)}$

E.4.5 *The open-loop transfer function of a unity feedback control system is given by $G(s)=K(s+2)/s(s-2)(s^2+5s+16)$. Determine the value of K which will cause sustained oscillations in the closed-loop system and what is the corresponding oscillation frequencies?*

E.4.6 *The open-loop transfer functions of certain unity feedback system are given below. In each case, sketch the Nyquist plot and determine the stability of the system.*

a) $G(s) = \dfrac{K(s+3)}{s(s-1)}$

b) $G(s) = \dfrac{-1}{2s(1-20s)}$

c) $G(s) = \dfrac{K(1+2s)}{s(1+s)(1+s+s^2)}$

d) $G(s) = \dfrac{K(s+5)(s+40)}{s^3(s+200)(s+1000)}$

e) $G(s) = \dfrac{K}{(s+1)(s+1.5)(s+2)}$

f) $G(s) = \dfrac{K}{s(s+ \quad ^2 + 2s + 2)}$

E.4.7 *Determine the phase margin and gain margin of the system with following transfer functions.*

a) $G(s) = \dfrac{2}{(s+1)^2}$ b) $\dfrac{20}{s(s+1)(s^2+2s+2)}$

E.4.8 *The open-loop transfer function of a unity feedback system is given by, $G(s)= K/s(1+0.5s)(1+s)$.*

a) Determine the value of K so that the gain margin of the system is 6 db.

b) Determine the value of K so that the phase margin of the system is 30°.

E.4.9 *The open-loop transfer functions of certain unity feedback systems are given below. Sketch the root locus of each system.*

a) $G(s) = \dfrac{K(s+2)}{(s+3)^2(s^2+2s+17)}$

b) $G(s) = \dfrac{K}{s(s+3)(s^2+2s+2)}$

c) $G(s) = \dfrac{K(s^2+2s+10)}{s(s+2)(s+4)}$

d) $G(s) = \dfrac{K(s+1)}{s^2(s+12)}$

e) $G(s) = \dfrac{K(s+4)}{s(s+0.5)(s+2)}$

f) $G(s) = \dfrac{K}{s(s^2+8s+20)}$

g) $G(s) = \dfrac{K}{s(s+2)(s^2+2s+2)}$

h) $G(s) = \dfrac{K(s^2+1)}{s(s+2)}$

E.4.10 *A unity feedback system has an open-loop transfer function, $G(s) = K/s(s^2+8s+32)$. Sketch the root locus and determine the dominant closed loop poles with ζ=0.5. Determine the value of K at this point.*

E.4.11 *Draw the root locus plot for a unity feedback system having forward path transfer function,*

$G(s) = K/s(s+1)(s+5)$.

a) Determine the value of K which gives continuous oscillations and the frequency of oscillation.

b) Determine the value of K corresponding to a dominant closed loop pole with damping ratio, ζ= 0.7.